Consider the following NFA $\mathcal{A} = (Q, \Sigma, \delta, q_0, F)$:

$Q = \{q_0, q_1, q_2\}$
$\Sigma = \{0, 1\}$
$F = \{q_2\}$

1) Describe the language accepted by this NFA.

2) Convert the NFA to a DFA using the subset construction.
   Keep only the essential states.

1) $L(\mathcal{A}) = \{w \in \{0, 1\}^* \mid w$ ends with a sequence of at least two 0's followed by only 1's $\}$

2)

$A_D = (Q_D, \Sigma, \delta_D, q_{0D}, F_D)$
with
$Q_D = 2^Q$
$q_{0D} = \{q_0\}$
$F_D = \{S \subseteq Q \mid F \cap S \neq \emptyset\}$
Consider the following DFA $A = (Q, \Sigma, \delta, q_0, F)$

Construct a regular expression $E_A$ s.t. $L(E_A) = L(A)$ by constructing the R.E. $E_{i,j}^k$ s.t.

$L(E_{i,j}^k) = \{w | A \text{ goes from } q_i \text{ to } q_j \text{ on } w \text{ passing only through } q_{k_1}, \ldots, q_{k_l}\}$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$E_{00}^k$</th>
<th>$E_{01}^k$</th>
<th>$E_{02}^k$</th>
<th>$E_{03}^k$</th>
<th>$E_{10}^k$</th>
<th>$E_{11}^k$</th>
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<th>$E_{31}^k$</th>
<th>$E_{32}^k$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>$\epsilon + 1$</td>
<td>0</td>
<td>$\emptyset$</td>
<td>1</td>
<td>$\epsilon$</td>
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<tr>
<td>1</td>
<td>$1^*$</td>
<td>$1^*0$</td>
<td>$\emptyset$</td>
<td>$1^*$</td>
<td>$\epsilon + 1^*0$</td>
<td>0</td>
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<td>0</td>
<td>$1^*$</td>
<td>$1^*0$</td>
<td>$3 + 0$</td>
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<tr>
<td>2</td>
<td>$1^*0$</td>
<td>$(1^<em>0)^</em>$</td>
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<td>$0^* + 3 + 0$</td>
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<td>3</td>
<td>$1^*0$</td>
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We use $E_{i,j}^k = E_{i,k}^{h-k} \cdot (E_{h,k}^{k-h})^* \cdot E_{h,j}^{h-k} + E_{i,j}^{k-h}$
Exercise:

Convert the following regular expression \( E \) to an \( \varepsilon \)-NFA, and then eliminate the \( \varepsilon \)-transitions:

\[ E = 1((00)^* + 1^*) \]

\[ \rightarrow 0 \rightarrow 0 \quad \text{accepts } 0 \]

\[ \rightarrow 0 \rightarrow 0 \rightarrow 0 \quad \text{accepts } 00 \quad (\text{we have merged initial and final states}) \]

\[ \rightarrow 0 \rightarrow 3 \rightarrow 0 \rightarrow 0 \rightarrow 3 \quad \text{accepts } 1^* \]

\[ \rightarrow 0 \rightarrow 3 \rightarrow 0 \rightarrow 0 \rightarrow 3 \rightarrow 0 \rightarrow 0 \rightarrow 3 \rightarrow 4 \rightarrow 0 \rightarrow 3 \quad \text{accepts } (00)^* + 1^* \quad (\text{we have merged initial and final states}) \]
To eliminate $\varepsilon$-transitions, remember that
\[
\delta_\varepsilon(q, \varepsilon) = \text{Id}(q) \cup \bigcup_{q' \in \text{Id}(q)} \delta(q', \varepsilon)
\]

A simpler automaton for $\mathcal{L}(E)$ is the following.
Exercise 4.2.3

Let $L$ be a language over $\Sigma$, and $e \in \Sigma$

We define $L/e = \{w \in \Sigma^* \mid we \in L\}$

Example: \(L = \{a, aeb, bae, bbe\}\)

\(L/e = \{\varepsilon, be, bb\}\)

$L/e$ is called the quotient of $L$ and $e$.

Show the following: If $L$ is regular, so is $L/e$, for $e \in \Sigma$.

(closure under quotient)

Proof:

Let $A_e = (Q, \Sigma, \delta, q_0, F_e)$ be a DFA s.t. $L(A_e) = L$.

We define $A_{L/e} = (Q, \Sigma, \delta, q_0, F_e)$ with

\[F_e = \{q \in Q \mid \delta(q, e) \in F\}\]

We show that $L(A_{L/e}) = L/e$

1) $L/e \subseteq L(A_{L/e})$

Let $w \varepsilon \in L$. Then \(\hat{\delta}(q_0, w, e) = \delta(\delta(q_0, w), e) \in F\).

Hence, by definition of $F_e$, \(\hat{\delta}(q_0, w) \in F_e\).

It follows that $w \varepsilon \in L(A_{L/e})$.

2) $L(A_{L/e}) \subseteq L/e$

Let $w \varepsilon \in L(A_{L/e})$. Then \(\hat{\delta}(q_0, w) = q \in F_e\).

Hence, by definition of $F_e$, $\delta(q, e) \in F$.

Hence, $\delta(q, e) = \delta(\hat{\delta}(q_0, w), e) = \delta(q_0, w, e) \in F$ and $we \varepsilon \in L$. It follows that $w \varepsilon \in L/e$. 

\[E \hat{7.3}\]
Exercise:

Minimize the following DFA

We construct the table of distinguishabilities, which corresponds to determine the equivalence classes w.r.t $\equiv_i$

$$
\begin{array}{cccccc}
  & A & B & C & D & E \\
 B & 0 & 0 & 0 & 0 & 0 \\
 C & 0 & 0 & 0 & 0 & 0 \\
 D & 0 & 1 & 0 & 1 & 0 \\
 E & 0 & 1 & 0 & 1 & 0 \\
 F & 0 & 1 & 0 & 1 & 1 \\
\end{array}
$$

$\equiv_0 : \{A, C\} \{B, D, E, F\}$

$\equiv_1 : \{A, C\} \{B\} \{E\} \{D, F\}$

$\equiv_2 = \equiv_1$
Exercise 5.11c

Define a context-free grammar that generates the set $L$ of all strings over \{a, b\} that are not of the form $ww$ (i.e., are not equal to any string repeated)

$$G = (V_N, V_T, P, S)$$

with

$$V_N = \{ S, A, B, C \}$$
$$V_T = \{ a, b \}$$

$$P:$$

$$S \rightarrow A \mid B \mid AB \mid BA$$
$$A \rightarrow CAC \mid a$$
$$B \rightarrow CBC \mid b$$
$$C \rightarrow e \mid b$$

Explanation:

- A generates an arbitrary odd length string with an e in the middle.

- Each odd length string is in $L$, hence $S \rightarrow A \mid B$

- Let $w$ be an even length string in $L$.

  Then $w = x.y$ with $|x| = |y|$, and $x$ and $y$ differ in at least one character: let $x = C^i a C^d$ (similarly for $y = C^i b C^d$). Each $C$ stands for either $a$ or $b$.

  Then: $w = x.y = C^i a C^d C^i b C^d = C^i a C^d b C^d = C^i a C^d C^d$.

  Hence $w$ is an odd length string with an $e$ in the middle.

- Hence $S \rightarrow AB$ (similarly, $S \rightarrow BA$)