Exercise

Write a grammar for the language \[ \{ a^n^2 | n \geq 1 \} \]

Solution

\[
S \rightarrow IT \\
T \rightarrow aTM | a \\
aM \rightarrow MAa \\
aA \rightarrow Aa \\
AM \rightarrow MA \\
IM \rightarrow I \\
IA \rightarrow aI \\
I \rightarrow e
\]

Comments:
I is a marker at the beginning of the strings; the second production generates \( n \) symbols \( a \) and \( n-1 \) symbols \( M \). The idea is that \( M \) is a sort of "multiplier", adding a symbol \( A \) for every symbol \( a \). At the end, the marker \( I \) "braces" from left to right, "killing" symbols \( M \) and turning \( A \) into \( a \). The number of symbols \( a \) at the end, being \( n \), times the number of \( A \) generated in the beginning, is \( m + (m-1)m = m^2 \).

Example of derivation:

\[
Exercise

Prove that the language $L = \{ a^k \mid k \text{ is not prime} \}$ is not regular.

Solution

We have already proved that the language

$$M = \{ a^k \mid k \text{ is prime} \}$$

is not regular (see page 4.3). By contradiction, assume that $L$ is regular. Since the class of regular languages is closed under complementation, we have that $\overline{L}$ is regular. Since $\overline{L} = M$, then $M$ is regular too, which is a contradiction. Therefore, $L$ cannot be regular.
Exercise

Give a context-free grammar for the language over \( \Sigma = \{0,1\} \) defined as follows: \( \{0^i 1^i 2^i \mid i \geq 2 \} \) and \( i \geq 2 \).

Solution

The grammar is

\[
S \rightarrow 0S11 \mid 0S1 \mid e
\]

Exercise

Prove that the language

\[
L = \{ a^{2^k} \mid k \geq 1 \}
\]

is not regular.

Solution

We apply the pumping lemma, assuming that \( L \) is regular. We choose \( w = a^{2^{m^2}} \) by choosing

\( w = a^{2^{m^2}} \) in fact here \( |w| = 2^{m^2} > m^2 \).

By the pumping lemma, \( w = xyz \), \( |xy| \leq m \), \( xy^2z \in L \) for all \( i \geq 0 \). We choose \( i = 2 \), so

\[ w' = xy^{2}z \in L. \]

Observe that \( |w'| = |w| + |y| = m^2 + |y| \). Since \( |xy| \leq m \), a fortiori \( |y| \leq m \), so

\[ |w'| = m^2 + 2 |y| \leq m^2 + 2m < m^3 + m^2 + 3m + 3 = (m+1)^3. \]

Therefore \( |w'| = m^2 + 2 |y| < (m+1)^3 \)

so \( |w'| \) cannot be a perfect cube.
Ex 6.4 (4.42 from textbook)

Minimize the following DFA.

Solution

We construct the list of distinguishability. Notice that, in the table, pairs marked with $i$ are those that are not equivalent with respect to $\equiv_i$ and are equivalent w.r.t. $\equiv_{i-1}$. In the beginning, we work with the pair $(q_1, q_2)$ where $q_1$ is fixed and $q_2$ is not or vice versa; in fact, we are determining classes of equivalence w.r.t. $\equiv_0$. At the next step we somehow refine the previous partitioning by determining pairs of states that are not equivalent w.r.t. $\equiv_1$ (we recall that if $q_1 \neq q_2$ then $q_1 \neq q_2)$, we proceed in this way until the partitioning cannot be further refined.
At the first step the marks "0" partition the set of states w.r.t. $\equiv_0$.

$$\{c,f\}, \{a,b,d,e,g,h,i\}$$

The partition w.r.t. $\equiv_1$ is the following:

$$\{c,f\}, \{a,b,d,e\}, \{g\}, \{h\}$$

Finally, the partition w.r.t. $\equiv_2$, obtained by considering marks "0", "1", and "2", is

$$\{a\}, \{b\}, \{c\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$$

Therefore the automaton is already minimal.