Exercise 2.2: Show that \( \forall q \in Q, \forall k, y \in \Sigma^* \):
\[
\hat{\delta}(q, k, y) = \hat{\delta}(\hat{\delta}(q, k), y)
\]

Solution: By induction on \(|y|\)

Base case: \(y = \varepsilon\)
\[
\hat{\delta}(q, k, \varepsilon) = \hat{\delta}(q, k) = \hat{\delta}(\hat{\delta}(q, k), \varepsilon) = \hat{\delta}(\hat{\delta}(q, k), y)
\]
since for any state \(q'\)
\[
\hat{\delta}(q', \varepsilon) = q'
\]

Inductive case: \(y = z \cdot a\)
\[
\hat{\delta}(q, k, y) = \hat{\delta}(q, k, z \cdot a) = \hat{\delta}(\hat{\delta}(q, k, z), a) = \hat{\delta}(\hat{\delta}(\hat{\delta}(q, k, z), a) = k [\text{Def of } \hat{\delta}] [\text{Inductive hypothesis}]
\]
\[
= \hat{\delta}(\hat{\delta}(\hat{\delta}(q, k), z), a) = \hat{\delta}(\hat{\delta}(\hat{\delta}(q, k), y)
\]

Exercise 2.2.5:

2) DFA that accepts \(\{w \in \{0, 1\}^* \mid \text{each consecutive block of 5 symbols contains at least two 0's} \}\)

![Diagram of DFA]

- \(\Rightarrow\) number of 1's without two 0's yet
- After e 1, zero 0's
- After e 1, one 0
Exercise 2.2.8:

Let $A$ be a DFA such that for some $q \in Q$ and all $q \in Q$
we have $\delta(q, a) = q$.

1) Show that for all $m > 0$, and all $q \in Q$, $\hat{\delta}(q, e^m) = q$.

2) Show that either $\{e\}^* \subseteq L(A)$ or $\{e\}^* \cap L(A) = \emptyset$.

Proof:

1) By induction on $m$.

- $m = 1$: $\hat{\delta}(q, e^1) = \delta(\delta(q, e), e) = \delta(q, e) = q$.

Suppose that for all $i < m$, $\hat{\delta}(q, e^i) = q$.

We show that also $\hat{\delta}(q, e^m) = q$.

$\hat{\delta}(q, e^m) = \hat{\delta}(\hat{\delta}(q, e^{m-1}), a) = \delta(q, a) = q$.

2) By part (a), we have that $\hat{\delta}(q_0, e^m) = q_0$, $\forall m > 0$.

If $q_0 \in F$, then $\hat{\delta}(q_0, e) = q_0 \in F$. Hence $e \in L(A)$.

Moreover, for $\forall m > 0$, we have $\hat{\delta}(q_0, e^m) = q_0 \in F$.

It follows that for all $m > 0$, $e^m \in L(A)$, i.e. $\{e\}^* \subseteq L(A)$.

If $q_0 \notin F$, then $\hat{\delta}(q_0, e) = q_0 \notin F$. Hence $e \notin L(A)$.

Moreover, for $\forall m > 0$, we have $\hat{\delta}(q_0, e^m) = q_0 \notin F$.

It follows that for all $m > 0$, $e^m \notin L(A)$, i.e. $\{e\}^* \cap L(A) = \emptyset$.

q.e.d.
Let $\mathcal{A} = (Q, \Sigma, S, q_0, \{q_f\})$ be a DFA. Let \( \delta(q_0, e) = \delta(q_f, e) \) for all $e \in \Sigma$

a) Show that for all $w \neq \varepsilon$, we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_f, w)$

b) Show that for all $x \in \mathcal{L}(\mathcal{A})$ with $k \neq \varepsilon$, we have $x^k \in \mathcal{L}(\mathcal{A})$, for all $k > 0$.

**Proof:**

b) By induction on $|w|$

\[ |w| = 1 \Rightarrow w = e \text{ for some } e \in \Sigma \]

\[ \hat{\delta}(q_0, w) = \hat{\delta}(q_0, e) = \hat{\delta}(q_f, e) = \hat{\delta}(q_f, w) \]

\[ |w| = m > 1 \Rightarrow \exists k \text{ s.t. } k < m \]

Let $w = x \cdot e$ with $|x| = m - 1$

\[ \hat{\delta}(q_0, w) = \hat{\delta}(q_0, x \cdot e) = \hat{\delta}(\hat{\delta}(q_0, x), e) = \hat{\delta}(q_f, x \cdot e) = \hat{\delta}(q_f, w) \]

b) By induction on $k$

\[ k = 1 \Rightarrow \text{statement is given by assumption } x = x^1 \in \mathcal{L}(\mathcal{A}) \]

\[ k > 1 \Rightarrow \text{assume that } x^k \in \mathcal{L}(\mathcal{A}) \]

\[ \hat{\delta}(q_0, x^k) = \hat{\delta}(q_0, x \cdot x^{k-1}, x) = \hat{\delta}(\hat{\delta}(q_0, x), x^{k-1}, x) = \hat{\delta}(q_f, x) = q_f \]

Hence $x^k \in \mathcal{L}(\mathcal{A})$.
Exercise 2.3.5:

Let $A_D = (Q, \Sigma, \delta_D, q_0, F)$ be a DFA and $A_N = (Q, \Sigma, \delta_N, q_0, F)$ be an NFA with $\delta_N(q, a) = \{p\} \iff \delta_D(q, a) = p$ for all $q \in Q$, $a \in \Sigma$.

Show that $\hat{\delta}_N(q_0, w) = \{\hat{\delta}_D(q_0, w)\}$ for all $w \in \Sigma^*$.

Proof: by induction on $|w|$

- $w = \varepsilon$:
  
  $\hat{\delta}_N(q_0, \varepsilon) = \{q_0\} = \{\hat{\delta}_D(q_0, \varepsilon)\}$

- Let $|w| = n + 1$ and assume the claim holds for all $\omega$ with $|\omega| \leq n$.

  Let $w = \omega \cdot a$, $\hat{\delta}_D(q_0, \omega) = q$ and $\delta_D(q, a) = p$.

  $\hat{\delta}_D(q_0, w) = \hat{\delta}_D(q_0, \omega \cdot a) = \hat{\delta}_D(\hat{\delta}_D(q_0, \omega), a) = \delta_N(q, a) = p$.

  By induction hypothesis, we have that

  $\hat{\delta}_N(q_0, \omega) = \hat{\delta}_D(q_0, \omega) = \{\hat{\delta}_D(q_0, \omega)\} = \{q\}$

  Hence $\hat{\delta}_N(q_0, w) = \hat{\delta}_N(q_0, \omega \cdot a) = \bigcup_{p \in \hat{\delta}_N(q_0, \omega)} \hat{\delta}_N(p, a) = \hat{\delta}_N(q, a) = p$.

  By definition of $\hat{\delta}_N$, and since $\delta_D(q, a) = p$,

  $\hat{\delta}_N(q, a) = \{p\} = \{\hat{\delta}_D(q_0, w)\}$
Exercise E2.2

In $k \geq 1$, define an NFA $A_N^k$ s.t.

$L(A_N^k) = \{ w \in \{0,1\}^* \mid \text{the } k\text{-th last symbol of } w \text{ is } e_k \}$

Exercise E2.3

For $\Sigma_k = \{ e_1, \ldots, e_k \}$, construct an NFA $A_N^k$ s.t.

$L(A_N^k) = \{ w \in \Sigma_k^* \mid w \text{ does not contain at least one of the symbols } e_1, \ldots, e_k \}$
Exercise 2.3.1:

Convert the following NFA to a DFA.

Exercise 2.3.4:

Give NFA's that accept the following languages over \{0,\ldots,9\}:

c) set of strings so the final digit has appeared before.

We use states \( q_i \), for \( i \in \{0,\ldots,8\} \) to guess that the final digit is \( i \).

d)