DL-Lite and Interval Temporal Logics: a Marriage Proposal (extended version)

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1 Introduction

Description logics [10] (DLs) are widely-used logical formalisms for knowledge representation, where the domain of interest is structured in concepts whose properties are specified by roles. Complex concepts and role expressions are constructed, starting from atomic ones, by applying suitable (logical) operators, whose availability depends on the specific language we consider. Concept descriptions are then collected into a *knowledge base* (KB), made of *intensional knowledge* (TBox assertions) and *extensional knowledge* (ABox assertions). A TBox typically consists of a set of axioms stating the inclusion between pairs of concepts or roles, while in an ABox one can assert membership of objects (constants) in concepts, or that a pair of objects is connected by a role.

The name *DL-Lite* identifies a family of description logics, proposed for the first time in [22, 23] and characterized by a good computational behaviour combined with a relatively high expressive power. To describe the *DL-Lite* family, whose intended applications include capturing typical concept modeling formalisms such as UML, class diagrams, and ER diagrams, we focus our attention on the *supremum* (w.r.t. the expressive power) formalism, namely, $\textit{DL-Lite}^{\mathcal{HN}}_{bool}$, which includes as a fragment every other element of the family. Among the various fragments, we mention sub-Boolean fragments (hence the subscript *bool*, which indicates no restrictions), fragments that cannot constrain the cardinality of roles, and fragments with restricted role inclusions in the TBox (hence the superscript $\mathcal{HN}$, which indicates that number restrictions and role inclusions are fully available). A comprehensive survey of the variety of the formalisms that belong to *DL-Lite*, their properties, and their applications can be found in [3].

Temporal extensions of *DLs* have been studied in [4, 6, 7, 9, 11, 12, 33] (see [5, 26, 29] for a detailed survey). In particular, in [8] logics of the family *DL-Lite* have been been combined with a variety of *point-based* temporal logics, ranging from LTL with Future and Past to full LTL with Since and Until (see, e.g., [27]). Different resulting logics, generically denoted by T_{TL}*DL-Lite*, are introduced according to the following parameters: *(i)* the fragment of LTL to be used, *(ii)* the fragment of *DL-Lite* taken as the basis for the extension, and *(iii)* whether roles can be temporalized or not. Complexities of the different logics range from NLogSpace to undecidable (in particular when roles can be temporalized or when role inclusions and number restrictions can interact without constraints). As for the underlying temporal structure, in [8] all results are given for $\mathbb{Z}$.

In this paper, we study *interval-based* extensions of *DL-Lite* logics based on fragments of Halpern and Shoham's logic $\mathcal{HS}$ [28]. We make the following assumptions: *(i)* we take $\mathbb{Z}$ as the temporal domain; *(ii)* we consider only $\mathcal{HS}$ fragments that can express the so-called *length constraints*; *(iii)* we distinguish between *rigid* roles, that is, roles that are time-invariant, and *flexible* roles, but we do not allow the computationally-expensive *temporalised* roles. All logics studied here can be considered as fragments of the full product of *DL-Lite*$^{\mathcal{HN}}_{bool}$ and $\mathcal{HS}$, that is, $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$. Since $\mathcal{HS}$ is undecidable over $\mathbb{Z}$, it easily follows that $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$ is undecidable as well. However, a number of recent results show that various fragments of $\mathcal{HS}$ offer a good balance between expressiveness and decidability/complexity [13, 14, 17, 19, 30, 31, 32], suggesting that we can weaken the temporal part, by considering decidable fragments of $\mathcal{HS}$, without sacrificing expressiveness.

Even if we fix the underlying interval-based temporal languages, there are a number of choices to be made. First of all, interval-based temporal logics have been studied, in a sense, in a more general way that point-based, as there is no immediate need to fix a class of linearly ordered sets right at the beginning. As a matter of fact, many interesting results have been found in the dense case or in the case of all linearly ordered sets (see, e.g., [24]). Second, we need to decide how to deal with ABox assertions. From [8] it seems clear that for useful applications the possibility of expressing that, for example, a certain individual has a given property *at a precise moment of time* cannot be given up. Fragments of point-based T_{TL}*DL-Lite* that do not include the Next operator, but only qualitative information (and therefore unable to directly represent ABox information), have been studied in [8], but the possibility of expressing ABox assertions is preserved by means of a clever technical solution.

2 The Language of *DL-Lite*

The language *DL-Lite*$^{\mathcal{HN}}_{bool}$ contains *object* names (a.k.a. *constants, individuals*) $a_0, a_1, \ldots$, *(atomic) concept* names $A_0, A_1, \ldots$, and *(atomic) role* names P_0, $P_1, \ldots$. Complex *concepts*, generically denoted by $C_0, C_1, \ldots$ as well as complex *roles*, denoted by $R_0, R_1, \ldots$ are formed by means of the following abstract grammar:

$$\begin{array}{rcl} R & ::= & P_k \mid P_k^-, \\ B & ::= & \bot \mid A_k \mid \ \geq q\ R, \\ C & ::= & B \mid \neg C \mid C_1 \sqcap C_2, \end{array}$$

where $q \geq 1$ and P_k^- denotes the *inverse* of the atomic role P_k. As in their logical counterpart, disjunction ($\sqcup$) and truthness ($\top$) are treated as standard abbreviations. $\exists R$ is an abbreviation for $\geq 1R$.

A *DL-Lite*$_{bool}^{\mathcal{HN}}$ TBox $\mathcal{T}$ is a finite set of concepts and roles *inclusions* of the form:

$$C_1 \sqsubseteq C_2, \text{ and } R_1 \sqsubseteq R_2,$$

respectively. Similarly, a *DL-Lite*$_{bool}^{\mathcal{HN}}$ ABox $\mathcal{A}$ is a finite set of *assertions* of the form:

$$A_k(a_m), \neg A_k(a_m), P_k(a_m, a_n), \neg P_k(a_m, a_n).$$

All together, $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$ is called a *knowledge base.*

An *interpretation* consists of a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty *domain* and $\cdot^{\mathcal{I}}$ is an interpretation function which assigns elements of $\Delta^{\mathcal{I}}$ to object names, subsets of $\Delta^{\mathcal{I}}$ to atomic concept names, and subsets of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to atomic role names. Under the *unique name assumption* (UNA), different object names are assigned to different domain elements. Complex concepts and roles are then interpreted as follows:

$$\begin{array}{lcl} (P_k^-)^{\mathcal{I}} & = & \{(y, x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x, y) \in P_k^{\mathcal{I}}\}, \\ \bot^{\mathcal{I}} & = & \emptyset, \\ (\geq q\ R)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \in \Delta^{\mathcal{I}} \mid (x, y) \in R^{\mathcal{I}}\} \geq q\}, \\ (\neg C)^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}, \\ (C_1 \sqcap C_2)^{\mathcal{I}} & = & C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}. \end{array}$$

An interpretation satisfies $C_1 \sqsubseteq C_2$ ($R_1 \sqsubseteq R_2$) if $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ ($R_1^{\mathcal{I}} \subseteq R_2^{\mathcal{I}}$) and it satisfies and assertion $A_k(a_m)$ ($\neg A_k(a_m), P_k(a_m, a_n), \neg P_k(a_m, a_n)$) if $a_m^{\mathcal{I}} \in A_k^{\mathcal{I}}$ ($a_m^{\mathcal{I}} \notin A_k^{\mathcal{I}}, (a_m^{\mathcal{I}}, a_n^{\mathcal{I}}) \in P_k^{\mathcal{I}}, (a_m^{\mathcal{I}}, a_n^{\mathcal{I}}) \notin P_k^{\mathcal{I}}$). We say that a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is *satisfiable* if there exists an interpretation $\mathcal{I}$ that satisfies every assertion of $\mathcal{K}$. Other interesting problems, such as *concept satisfiability* (i.e., deciding whether there exists an interpretation $\mathcal{I}$ of $\mathcal{K}$ such that a concept $C^{\mathcal{I}} \neq \emptyset$), or *subsumption* (deciding whether every interpretation $\mathcal{I}$ of $\mathcal{K}$ is such that $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$) can be reduced to KB satisfiability.

There are several ways that have been studied to reduce the expressive power of *DL-Lite*$_{bool}^{\mathcal{HN}}$ with the aim of reducing the complexity of its satisfiability problem. These techniques can be described as reductions along three axes: (i) restricting the applicability of boolean operators, (ii) eliminating the number restrictions, and (iii) restricting or eliminating the applicability of role inclusions as axioms of the TBox. The standard notation includes a subscript $\alpha \in \{bool, Horn, Krom, core\}$, and a superscript

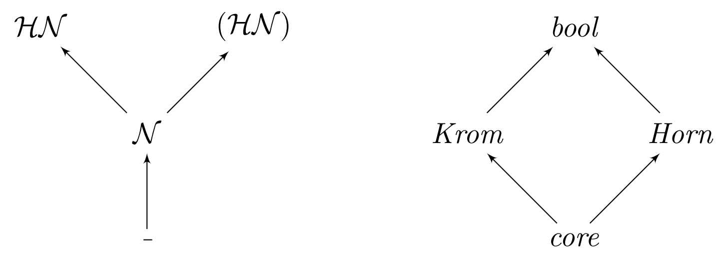

Figure 1: Relative expressive power between elements of the *DL-Lite* family.

$\beta \in \{\_, \mathcal{N}, \mathcal{H}, \mathcal{HN}, (\mathcal{HN})\}$ to denote the choice. In particular, including $\mathcal{N}$ (resp., $\mathcal{H}$) denotes that number restrictions (resp., role inclusions) are fully available, and $(\mathcal{HN})$ indicates that both number restrictions and role inclusions are allowed, but no role R can occur in the TBox in both a role inclusion and a number restriction with $q \geq 2$. Conversely, *bool* denotes the fact that Boolean operators are not restricted, while *Horn* (resp., *Krom,core*) indicate that Boolean operators are limited as in the corresponding fragment of propositional logic; this limitation applies to the construction of TBox axioms. From an expressive power point of view, the various limitations relate to each other independently, giving rise to two (partial) diagrams as in Fig. 1, and from the complexity point of view the obtained fragments range from NLogSpace (sub-boolean fragments) to NP and to ExpTime (when role inclusions and number restrictions can be freely used).

3 The Interval Temporal Logic $\mathcal{HS}$ and its Fragments

Let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set. An *interval* over $\mathbb{D}$ is an ordered pair $[i, j]$, where $i, j \in D$ and $i < j$ (*strict semantics*). There are 12 different non-trivial ordering relations (excluding equality) between any pair of intervals in a linear order, often called *Allen's relations* [2]: the six relations depicted in Fig. 2 and the inverse ones. We interpret interval structures as Kripke structures and Allen's relations as accessibility relations, thus associating a modality $\langle X \rangle$ with each Allen's relation R_X. For each operator $\langle X \rangle$, its *inverse* (or *transpose*), denoted by $\langle \overline{X} \rangle$, corresponds to the inverse relation $R_{\overline{X}}$ of R_X (that is, $R_{\overline{X}} = (R_X)^{-1}$). Halpern and Shoham's logic $\mathcal{HS}$ is a multi-modal logic with formulas built on a set $\mathcal{AP}$ of proposition letters, the boolean connectives $\vee$ and $\neg$, and a modality for each Allen's relation. We denote by $\mathsf{X_1} \ldots \mathsf{X_k}$ the fragment of $\mathcal{HS}$ featuring a modality for each Allen's relation in the subset $\{R_{X_1}, \ldots, R_{X_k}\}$. Formulas of $\mathsf{X_1} \ldots \mathsf{X_k}$ are defined by the grammar:

$$\varphi ::= p \mid \neg\psi \mid \psi \vee \tau \mid \langle X_1 \rangle \psi \mid \ldots \mid \langle X_k \rangle \psi.$$

$\langle A \rangle$	$[i,j]R_A[i',j'] \Leftrightarrow j = i'$
$\langle L \rangle$	$[i,j]R_L[i',j'] \Leftrightarrow j < i'$
$\langle B \rangle$	$[i,j]R_B[i',j'] \Leftrightarrow i = i', j < j'$
$\langle E \rangle$	$[i,j]R_E[i',j'] \Leftrightarrow j = j', i < i'$
$\langle D \rangle$	$[i,j]R_D[i',j'] \Leftrightarrow i < i', j' < j$
$\langle O \rangle$	$[i,j]R_O[i',j'] \Leftrightarrow i < i' < j < j'$

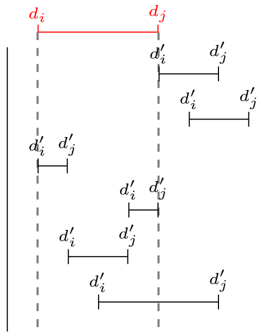

Figure 2: Allen's interval relations and the corresponding $\mathcal{HS}$ modalities.

The other boolean connectives can be viewed as abbreviations, and the dual operators $[X]$ are defined as usual. The semantics of $\mathcal{HS}$ is given in terms of *interval models* $\mathcal{M} = \langle \mathbb{I}(\mathbb{D}), \mathcal{V} \rangle$, where $\mathbb{I}(\mathbb{D})$ is the set of all intervals over $\mathbb{D}$ and $\mathcal{V} : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{D})}$ is a *valuation function* that assigns to every $p \in \mathcal{AP}$ the set of intervals $\mathcal{V}(p)$ over which p holds. The *truth* of a formula over a given interval $[i,j]$ in an interval model $\mathcal{M}$ is defined by structural induction on formulas:

$$
\begin{array}{lll}
\mathcal{M}, [i,j] \Vdash p & \text{iff} & [i,j] \in \mathcal{V}(p) \\
\mathcal{M}, [i,j] \Vdash \neg\psi & \text{iff} & \mathcal{M}, [i,j] \not\Vdash \psi \\
\mathcal{M}, [i,j] \Vdash \psi \wedge \tau & \text{iff} & \mathcal{M}, [i,j] \Vdash \psi \text{ and } \mathcal{M}, [i,j] \Vdash \varphi \\
\mathcal{M}, [i,j] \Vdash \langle X_k \rangle\psi & \text{iff} & \mathcal{M}, [i',j'] \Vdash \psi \text{ for some } [i,j]R_X[i',j'].
\end{array}
$$

Recently, a great effort has been devoted to the study of decidability of fragments of $\mathcal{HS}$. Ever since $\mathcal{HS}$ was introduced, it was immediately clear that its satisfiability problem is undecidable when interpreted on every interesting class of linearly ordered sets [28]. While this sweeping result initially discouraged further research in this direction, recent results showed that the situation is slightly better then it seemed. Given the set of $\mathcal{HS}$ modalities that correspond to the set of Allen's relations $\{R_{X_1}, \ldots, R_{X_k}\}$, we call *fragment* $\mathcal{F} = \mathsf{X_1X_2 \ldots X_n}$ any subset of such modalities, displayed in alphabetical order. There are 2^{12} such fragments. Some of these are expressively equivalent to each other; in [24] (resp., [1]) it is possible to find all possible inter-definability in the class of all linearly ordered sets (resp., all dense linearly ordered sets), giving rise to 1347 (resp., 966) expressively different fragments. The number of different fragments on other classes of linear orders has not been determined yet, but it is believed that the situation in the finite or discrete case should be similar. Out of these fragments, it has been possible to prove that exactly 62 are decidable in the finite case [15], and 44 in the (strongly) discrete case (and in the case of $\mathbb{Z}$) [16], all of which with complexities that range from NP-complete (in very simple cases) to NExpTime-complete, ExpSpace-complete, to non-primitive recursive. The complete diagram that includes all decidable fragments of $\mathcal{HS}$,

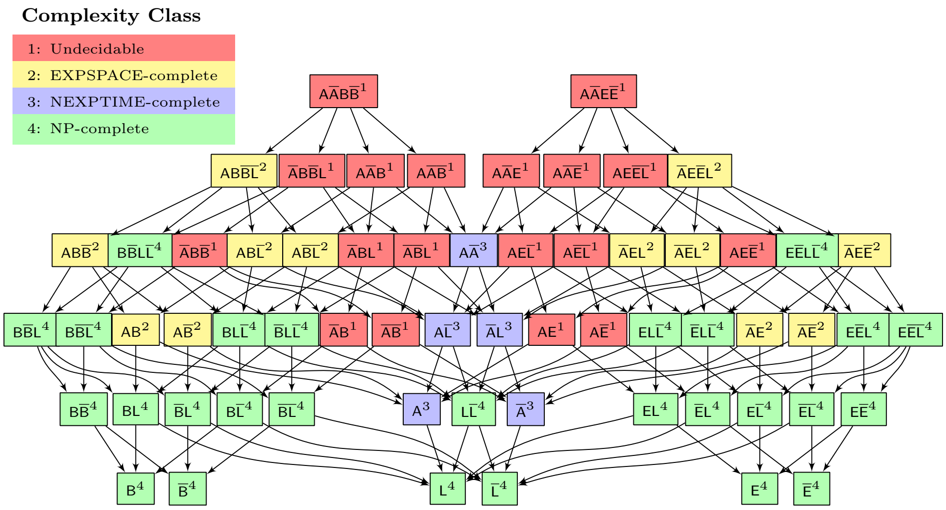

Figure 3: Hasse diagram of all decidable fragments $\mathcal{HS}$ over $\mathbb{Z}$.

along with their relative expressive power, over the integers, is displayed in Fig. 3. The way in which switching from $\mathbb{Z}$ to $\mathbb{N}$ influences the computational properties of fragments of $\mathcal{HS}$ is displayed in Fig. 4, which contains all decidable fragments of $\mathcal{HS}$ over natural numbers. The most interesting change is that some fragments, undecidable over $\mathbb{Z}$, become decidable, but non-primitive recursive.

Since extending *DL-Lite* with an interval-based temporal logics in search for a decidable hybrid has any hope to be successful only when the propositional interval-based logic alone is, at least, decidable, the diagrams in Fig. 3 and in Fig. 4 contain all fragments of $\mathcal{HS}$ in which we can be interested. In this paper we are interested in the set of the integers as backbone for the temporal part; changing this assumption, however, changes the decidability status of fragments of $\mathcal{HS}$ only in a few cases.

We now consider two expressivity issues relevant for obtaining complexity results on the considered interval-based *DL-Lite* logics, i.e., (i) the possibility of expressing the *universal* modality; (ii) the possibility of correctly expressing *length constraints* over the intervals. We show how to capture both of them when formulas are interpreted over $\mathbb{Z}$. $[G]$ is said to be a *universal* modality when:

$$\mathcal{M}, [i,j] \models [G]\varphi \quad \textit{iff} \quad \forall [i',j'] \in \mathbb{I}(\mathbb{Z}), \mathcal{M}, [i',j'] \models \varphi.$$

Whenever a fragment $\mathcal{F}$ of $\mathcal{HS}$ is powerful enough to express $[G]$, we have that its satisfiability problem on $\mathbb{Z}$ can be safely reduced to *initial satisfiability*, that is, over the interval $[0,1]$. In particular, in the fragment $\mathrm{A}\overline{\mathrm{A}}$, a possible way to express $[G]$ over $\mathbb{Z}$ is:

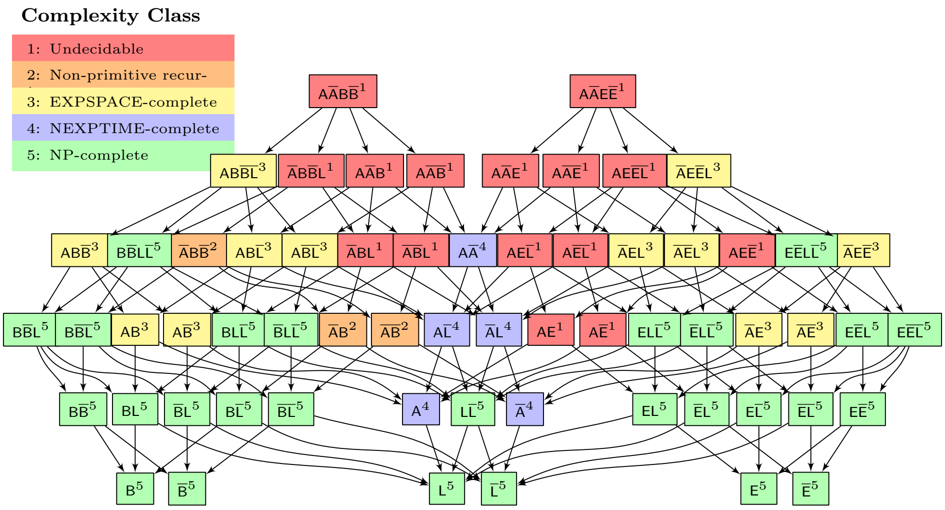

Figure 4: Hasse diagram of all decidable fragments $\mathcal{HS}$ over $\mathbb{N}$.

$$[G]\varphi \equiv [\overline{A}][\overline{A}][A]\varphi \wedge [\overline{A}][A]\varphi \wedge [\overline{A}][A][A]\varphi. \tag{1}$$

While the above expression is applicable in most of the fragments of Fig. 4 and Fig. 4, some of them are left out. These can be recovered with other (more complex) formulas, provided that they allow one to move in both directions. When this is not the case, then some intervals cannot be seen, and it is convenient to change the underlying class of linear orders to recover the universal modality.

Concerning the possibility to express *length constraints*, the situation is slightly more complex. The most general way to deal with length constraints is to explicitly introduce a predicate, $\mathsf{len}_{\sim k}$, with the following semantics [18]:

$$\mathcal{M}, [i, j] \models \mathsf{len}_{\sim k} \text{ iff } \delta(i, j) \sim k.$$

where δ is a *distance* function $\delta : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{N}$, defined as $\delta(i, j) = |i - j|$, for each $\sim \in \{<, \leq, =, \geq, >\}$[1]. As showed in [18, 20], the language of $A\overline{A}$ can be extended with explicit length constraints predicates when interpreted over $\mathbb{N}$ or $\mathbb{Z}$ without loosing the decidability of the fragment itself; its complexity, though, worsen from NExpTime to ExpSpace. Length constraints can also be captured inside $\mathcal{HS}$. The simplest way to achieve this is to make use of either $\langle B\rangle$ or $\langle E\rangle$ modality. For example, under the discreteness hypothesis, we have that:

$$\mathcal{M}, [i, j] \models \mathsf{len}_{=k} \;\; \textit{iff} \;\; \mathcal{M}, [i, j] \models \langle B\rangle^{k-1}\top \wedge [B]^{k}\bot.$$

[1] Equality and inequality constraints are mutually definable, although there is a increase in formula length if we consider only constraints of the form $\mathsf{len}_{=k}$ as primitive.

where we use the expression $\langle X_k \rangle^k$ to denote the application of a modality $\langle X_k \rangle$ k times. It is worth observing that the above encoding is *unary*: this means that a formula that contains a length constraint with a constant k, expressed in binary, will be *exponentially* longer than the constraint itself. To overcome this problem, we can use a more complicate *logarithmic* encoding (see [21]) that requires the use of both $\langle A \rangle$ and one among $\{\langle B \rangle, \langle \overline{B} \rangle, \langle E \rangle, \langle \overline{E} \rangle\}$. In other cases, as showed in [18, 20] for the fragment A$\overline{\text{A}}$, the language can be extended with length constraints without loosing the decidability of the fragment itself. This addition, though, worsen the complexity from NExpTime to ExpSpace.

The picture that can be drown from [16, 20] shows that, over $\mathbb{Z}$, two fragments are particularly interesting for us, namely $\mathsf{AB\overline{BL}}$ and $\mathsf{A\overline{A}}$ extended with length constraints (also known as MPNL). These are maximally decidable (in the sense that no other modality from the $\mathcal{HS}$ machinery can be added without loosing their decidability), both can express length constraints and the universal operator, and are expressively incomparable to each other. The decidability problem for both of them is ExpSpace-complete.

4 The Language $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$

The purpose of this paper is to define a family of languages that can be obtained by combining *DL-Lite* with $\mathcal{HS}$. These languages will have the possibility of describing concepts with characteristics that change over time and properties that hold over intervals instead of points. We present here the most expressive member of this family, that is, $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$.

The syntax of $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$ can be naturally obtained from the two components *DL-Lite*$^{\mathcal{HN}}_{bool}$ and $\mathcal{HS}$. Its language contains *object names* $a_0, a_1, \ldots$, *concept names* $A_0, A_1, \ldots$, *flexible role names* P_0, $P_1, \ldots$, and *rigid role names* $G_0, G_1, \ldots$ *Role names* S, *roles* R, *basic concepts* B and *concepts* C are formed by means of the following abstract grammar:

$$\begin{array}{lcl} S & ::= & P_k \mid G_k, \\ R & ::= & S \mid S^-, \\ B & ::= & \bot \mid A_k \mid \ \geq q\ R, \\ C & ::= & B \mid \neg C \mid C_1 \sqcap C_2 \mid \langle X_k \rangle C, \end{array}$$

where $\langle X_k \rangle$ is one of the $\mathcal{HS}$-modalities explained in the previous section and $[X_k]C \equiv \neg\langle X_k \rangle \neg C$. Members of the TBox can be built as in the atemporal case, while members of ABox are of the following form:

$$\begin{array}{ll} A_k(a_m, [i,j]), & \neg A_k(a_m, [i,j]), \\ S_k(a_m, a_n, [i,j]), & \neg S_k(a_m, a_n, [i,j]). \end{array}$$

A *temporal interpretation* for a $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$ knowledge base is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}([i,j])})$ where $\Delta^{\mathcal{I}} \neq \emptyset$ is a non empty domain (under the *constant* *domain assumption*) and $\mathcal{I}([i,j])$ is a standard *DL* interpretation for each interval $[i,j] \in \mathbb{I}(\mathbb{Z})$ such that:

$$\mathcal{I}([i,j]) = \langle \Delta^{\mathcal{I}}, a_0^{\mathcal{I}}, \dots A_0^{\mathcal{I}([i,j])}, \dots, P_0^{\mathcal{I}([i,j])}, \dots, G_0^{\mathcal{I}}, \dots \rangle$$

We assume that rigid roles names and object names have a time-invariant interpretation. Thus we omit the interval parameter in their interpretation, i.e., $G^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. The interpretation of flexible roles names $P^{\mathcal{I}([i,j])} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and concept names $A^{\mathcal{I}([i,j])} \subseteq \Delta^{\mathcal{I}}$ depend on the interval $[i,j]$ of evaluation. Interpreting atemporal concepts can be accomplished as before, while we interpret temporal concepts as:

$$(\langle X_k \rangle C)^{\mathcal{I}([i,j])} = \bigcup_{[i,j] r_{X_k} [i',j']} C^{\mathcal{I}([i',j'])},$$

where r_{X_k} is the Allen's relation that corresponds to the modality $\langle X_k \rangle$. Members of the TBox are interpreted *globally*:

$$\begin{aligned} \mathcal{I} \Vdash C_1 \sqsubseteq C_2 \quad &\text{iff} \quad C_1^{\mathcal{I}([i,j])} \subseteq C_2^{\mathcal{I}([i,j])}, \text{ for all } [i,j] \in \mathbb{I}(\mathbb{Z}); \\ \mathcal{I} \Vdash R_1 \sqsubseteq R_2 \quad &\text{iff} \quad R_1^{\mathcal{I}([i,j])} \subseteq R_2^{\mathcal{I}([i,j])}, \text{ for all } [i,j] \in \mathbb{I}(\mathbb{Z}). \end{aligned}$$

ABox assertions are interpreted *locally* on some interval $[i,j]$:

$$\begin{aligned} \mathcal{I} \Vdash A_k(a_m, [i,j]) \quad &\text{iff} \quad a_m \in A_k^{\mathcal{I}([i,j])}, \\ \mathcal{I} \Vdash \neg A_k(a_m, [i,j]) \quad &\text{iff} \quad a_m \notin A_k^{\mathcal{I}([i,j])}, \\ \mathcal{I} \Vdash S_k(a_m, a_n, [i,j]) \quad &\text{iff} \quad (a_m, a_n) \in S_k^{\mathcal{I}([i,j])}, \\ \mathcal{I} \Vdash \neg S_k(a_m, a_n, [i,j]) \quad &\text{iff} \quad (a_m, a_n) \notin S_k^{\mathcal{I}([i,j])}. \end{aligned}$$

The notion of satisfiability of a $T_{\mathcal{HS}}$*DL-Lite*$_{bool}^{\mathcal{HN}}$ knowledge base can be defined as follows.

Definition 1 *Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be a $T_{\mathcal{HS}}$DL-Lite$_{bool}^{\mathcal{HN}}$ knowledge base. Then, $\mathcal{K}$ is* satisfiable *($\mathcal{I} \models \mathcal{K}$) if and only if there exists an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}([i,j])})$ that satisfies every assertion in $\mathcal{A}$ and all axioms in $\mathcal{T}$.*

From the decidability point of view, we already know that $\mathcal{HS}$ alone is undecidable (over almost any meaningful class of linear orders, including $\mathbb{Z}$), and that the point-based temporal counterpart of *DL-Lite*$_{bool}^{\mathcal{HN}}$ (where number restrictions and role inclusions can be used freely) is undecidable as well when rigid roles are also allowed. Therefore, in the rest of the paper we focus our attention on how to combine decidable fragments of $\mathcal{HS}$ with a suitable fragment of *DL-Lite*$_{bool}^{\mathcal{HN}}$ to obtain a decidable logic.

We conclude this section with a small but meaningful example. Let us consider the ER diagram of Fig. 5 representing a small part of a medical

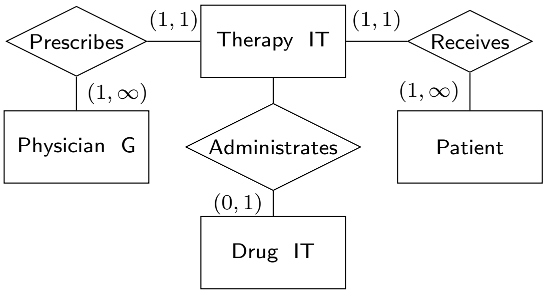

Figure 5: The conceptual data model of the medical example.

information system. The diagram represents an entity *Therapy* that is prescribed by some *Physician* to some *Patient*. A *Therapy* consists of the administration of some *Drug*. We can model the diagram in $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$ by considering *Therapy*, *Physician*, *Patient* and *Drug* as concept names, and *Prescribes*, *Receives* and *Administrates* as role names. Cardinality constraints on relationships, like *'a therapy is prescribed by exactly one physician'*, can be expressed as in *DL-Lite*:

$$Therapy \sqsubseteq \exists Prescribes^- \qquad \geq 2\ Prescribes^- \sqsubseteq \bot$$

The fact that *Physician* is a *time-invariant* entity — i.e., a physician is a global entity holding at every interval — is represented with the *timestamp* G (standing for *global*) and is enforced by using the global temporal operator and the corresponding axiom:

$$Physician \sqsubseteq [G]Physician$$

Time-variant entities are represented by marking them with the *timestamp* IT with the meaning that they have a limited life-span. The fact that *Therapy* and *Drug* are both time-variant can be captured using the following axioms:

$$Therapy \sqsubseteq \langle \overline{A} \rangle \neg Therapy \sqcap \langle A \rangle \neg Therapy,$$
$$Drug \sqsubseteq \langle \overline{A} \rangle \neg Drug \sqcap \langle A \rangle \neg Drug.$$

The interval modalities of $\mathcal{HS}$allow us to express complex relations between the lifespan of time-variant entities. For instance, we can say that *'a therapy starts and finishes with a drug administration'*:

$$Therapy \sqsubseteq \langle B \rangle \exists Administrates \sqcap \langle E \rangle \exists Administrates.$$

and that *'drug administrations cannot overlap inside the same therapy'*:

$$\exists Administrates \sqsubseteq [O] \neg \exists Administrates.$$

The above example makes use of $\mathcal{HS}$modalities that belong to fragments which are not always decidable. Under reasonable assumptions, such as, for example, *'a therapy is always shorter than k time units'*, we can rewrite the above formulas using only the interval modalities of the decidable fragments MPNL or $\mathsf{AB\overline{BL}}$.

5 Decidable Fragments of $T_{\mathcal{HS}}$DL-Lite$^{\mathcal{HN}}_{bool}$

To obtain decidable fragments of $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$ we investigate here the case where decidable fragments of $\mathcal{HS}$ are combined with *DL-Lite*$^{(\mathcal{HN})}_{bool}$, i.e., the restriction of *DL-Lite*$^{\mathcal{HN}}_{bool}$ with the following condition: if a role R appears in an axiom of the form $R' \sqsubseteq R$, for some role $R' \neq R$, then number restrictions $\geq q\ R$ and $\geq q\ R^-$, with $q \geq 2$, cannot appear in $\mathcal{T}$. Since, as showed in [3], the language *DL-Lite*$^{(\mathcal{HN})}_{bool}$ behaves computationally as *DL-Lite*$^{\mathcal{N}}_{bool}$, in the following we consider the language $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{N}}_{bool}$. Complexity results will then transfer to $T_{\mathcal{HS}}$*DL-Lite*$^{(\mathcal{HN})}_{bool}$.

Using a technique similar to [8], we first prove that $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ can be embedded into $\mathrm{FO}^1 \times \mathcal{HS}$, that is, $\mathcal{HS}$ extended with the single variable fragment of first order logic, and from that, into propositional $\mathcal{HS}$. The embedding can be accomplished for temporal fragments of $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ as well under the following conditions: (i) the universal operator is expressible; (ii) length constraints are expressible or explicitly added to the language; (iii) the interpretation is based on a left-bounded domain only if no past operators are present (notice that past operators in $\mathcal{HS}$ are: $\langle\overline{A}\rangle, \langle\overline{L}\rangle, \langle\overline{E}\rangle, \langle\overline{O}\rangle, \langle\overline{D}\rangle$). Whenever the fragment $\mathcal{F}$ of $\mathcal{HS}$ is decidable, and yet the above conditions are met, we prove that the obtained $T_{\mathcal{F}}$*DL-Lite* is decidable as well.

5.1 The language $\mathrm{FO}^1 \times \mathcal{HS}$

To the best of our knowledge, no first-order extensions of $\mathcal{HS}$ can be found in the literature. This is not surprising, given that $\mathcal{HS}$is already undecidable, that decidable fragments of it has been found only recently, and that even a very small first-order extension of a decidable fragment of $\mathcal{HS}$ has already been found to be undecidable [25].

The language of $\mathrm{FO} \times \mathcal{HS}$ contains *predicates* $P_0, P_1, \ldots$ of some given arity, *variables* $x_0, x_1, \ldots$, and constants $a_0, a_1, \ldots$ A *formula* of $\mathrm{FO} \times \mathcal{HS}$, generically denoted by $f, g, \ldots$, is built from the basic components by means of the following abstract grammar:

$$f ::= P_k(t_0, \ldots, t_s) \mid \bot \mid \forall x g \mid \neg g \mid g \wedge h \mid \langle X_k \rangle g,$$

where $t_0, t_1, \ldots$ are *terms* (i.e., variables or constants), P_k is of arity s, and $\langle X_k \rangle$ is one of the $\mathcal{HS}$ modalities. The existential quantification, the remaining Boolean operators and the universal temporal modalities can all be considered as shortcuts.

As in propositional $\mathcal{HS}$, we let $\mathbb{D} = \langle D, < \rangle$ be a linearly ordered set, and we consider the set of all intervals $\mathbb{I}(\mathbb{D})$ that can be built on it. Then, given a domain $\Delta^{\mathcal{M}} \neq \emptyset$, for each interval $[i, j]$ in $\mathbb{I}(\mathbb{D})$ we define the structure:

$$\mathcal{M}([i, j]) = \langle \Delta^{\mathcal{M}}, a_0^{\mathcal{M}}, \ldots, P_0^{\mathcal{M}([i,j])}, \ldots \rangle.$$

As before, under the rigid and constant domain assumption, the domain and the interpretation of constants do not vary over time. Therefore, a $\mathrm{FO} \times \mathcal{HS}$ *model* based on $\mathbb{D}$ is a tuple of the type $\mathcal{M} = (\Delta^{\mathcal{M}}, \mathcal{M}(\cdot))$, where $\mathcal{M}(\cdot)$ assigns to every interval in $\mathbb{I}(\mathbb{D})$ a first-order structure as described above. An *assignment* $\mathfrak{a}$ maps variables to elements of $\Delta^{\mathcal{M}}$. If $\mathfrak{a}$ and $\mathfrak{a}'$ are two assignments that differ exactly on the value assigned to the variable x, we write $\mathfrak{a}' \neq_x \mathfrak{a}$. The *truth* of a formula over a given interval $[i, j]$ and for a given assignment $\mathfrak{a}$ is defined by structural induction:

$$\begin{array}{lll}
\mathcal{M}([i,j]) \Vdash^{\mathfrak{a}} P_k(t_1, \ldots, t_s) & \text{iff} & (t_1^{\mathcal{M}}, \ldots, t_s^{\mathcal{M}}) \in P_k^{\mathcal{M}([i,j])} \\
\mathcal{M}([i,j]) \Vdash^{\mathfrak{a}} \neg g & \text{iff} & \mathcal{M}([i,j]) \not\Vdash^{\mathfrak{a}} g \\
\mathcal{M}([i,j]) \Vdash^{\mathfrak{a}} g \wedge h & \text{iff} & \mathcal{M}([i,j]) \Vdash^{\mathfrak{a}} g \text{ and} \\
 & & \mathcal{M}([i,j]) \Vdash^{\mathfrak{a}} h \\
\mathcal{M}([i,j]) \Vdash^{\mathfrak{a}} \forall x g & \text{iff} & \mathcal{M}([i,j]) \Vdash^{\mathfrak{a}'} g \text{ for all} \\
 & & \mathfrak{a}' \neq_x \mathfrak{a} \\
\mathcal{M}([i,j]) \Vdash^{\mathfrak{a}} \langle X_k \rangle g & \text{iff} & \mathcal{M}([i',j']) \Vdash^{\mathfrak{a}} \psi \text{ for some} \\
 & & [i',j'] \text{ s.t. } [i,j] r_{X_k} [i',j'].
\end{array}$$

For our purposes, we focus on $\mathrm{FO}^1 \times \mathcal{HS}$ (i.e., the one variable fragment of $\mathrm{FO} \times \mathcal{HS}$) interpreted over the integers and where satisfiability is intended as *initial* satisfiability (i.e., over $[0, 1]$) as in $\mathcal{HS}$.

Definition 2 *Let f be a $\mathrm{FO}^1 \times \mathcal{HS}$ formula. Then, f is* satisfiable *over Z if and only if there exists a model $\mathcal{M} = (\Delta^{\mathcal{M}}, \mathcal{M}(\cdot))$ based on $\mathbb{Z}$ and an assignment $\mathfrak{a}$ such that $\mathcal{M}, [0, 1] \Vdash^{\mathfrak{a}} f$.*

5.2 Embedding in $\mathrm{FO}^1 \times \mathcal{HS}$

We now show the main result of this paper. We consider $T_{\mathcal{HS}}$*DL-Lite*$_{bool}^{\mathcal{N}}$ KBs and we give an equi-satisfiable encoding into $\mathrm{FO}^1 \times \mathcal{HS}$ formulas. As a matter of facts, all proofs and results immediately transfer from $T_{\mathcal{HS}}$*DL-Lite*$_{bool}^{\mathcal{N}}$ to $T_{\mathcal{HS}}$*DL-Lite*$_{bool}^{(\mathcal{HN})}$.

Let us fix a $T_{\mathcal{HS}}$*DL-Lite*$_{bool}^{\mathcal{N}}$ knowledge base $\mathcal{K} = \mathcal{T} \cup \mathcal{A}$, where $\mathcal{T}$ contains only concept inclusions (role inclusions are not allowed in $T_{\mathcal{HS}}$*DL-Lite*$_{bool}^{\mathcal{N}}$), and let $\mathsf{ob}(\mathcal{A})$ be the set of objects in $\mathcal{A}$. Also, let us denote by $role(\mathcal{K})$ (resp., $role^r(\mathcal{K})$) the set of all role names (resp., rigid role names) in $\mathcal{K}$ plus their inverses, and by $Q_{\mathcal{K}}$ the set containing 1 and all numbers q s.t.

$\geq q\ R$ occurs in $\mathcal{K}$. Finally, for every role name R, we denote by $inv(R)$ its inverse, i.e., $inv(S) = S^-$ and $inv(S^-) = S$, for a role name S. In our encoding, objects in $\mathsf{ob}(\mathcal{A})$ become constants, and atomic concepts, A, and number restrictions, $\geq q\ R$, become unary predicates ($A(x)$ and $E_qR(x)$, respectively; the shortcut ER is used instead of E_1R). Intuitively, for a role name S, the predicates $E_qS(x)$ and $E_qS^-(x)$ represent the domain and the codomain of S. The $\mathrm{FO}^1 \times \mathcal{HS}$ formula $C^*(x)$ that encodes a $T_{\mathcal{HS}}DL\text{-}Lite^{\mathcal{N}}_{bool}$ concept C is built by induction on the structure of C:

$$
\begin{array}{lll}
A^* = A(x), & \bot^* = \bot, & (\geq q\ R)^* = E_qR(x) \\
(C_1 \sqcap C_2)^* = C_1^* \wedge C_2^*, & (\neg C)^* = \neg C^*, & (\langle X \rangle C)^* = \langle X_k \rangle C_k^*.
\end{array}
$$

To correctly translate $\mathcal{K}$ we have to describe how to deal with axioms and assertions. Translating an axiom $C_1 \sqsubseteq C_2$ requires a global formula. To this end, we can use the definable modal operator $[G]$ discussed in Section 3. Furthermore, we need to add conditions to correctly encode roles. First, if an object has q R-successors, then, it has also q' R-successors for any $q' \in Q_{\mathcal{K}}, q' < q$. Second, if the domain of a role is non-empty, then its range is not empty as well. Third, if a role R is rigid and an object has q R-successors at an interval $[i, j]$, then, it has q R-successors at every interval of the model. The translation $\mathcal{T}^\dagger$ of $\mathcal{T}$ is is defined as the conjunction of the following sentences:

$$\bigwedge_{C_1 \sqsubseteq C_2 \in \mathcal{T}} [G]\forall x(C_1^*(x) \rightarrow C_2^*(x)) \wedge \tag{2}$$

$$\bigwedge_{R \in role(\mathcal{K})} \bigwedge_{q, q' \in Q_{\mathcal{K}}, q' < q} [G]\forall x(E_qR(x) \rightarrow E_{q'}R(x)) \wedge \tag{3}$$

$$\bigwedge_{R \in role(\mathcal{K})} [G](\exists x ER(x) \rightarrow \exists x(E\,inv(R)(x))) \wedge \tag{4}$$

$$\bigwedge_{R \in role^r(\mathcal{K})} \bigwedge_{q \in Q_{\mathcal{K}}} [G]\forall x(E_qR(x) \rightarrow [G]E_qR(x)). \tag{5}$$

To complete the transformation, we need to deal with assertions in the ABox $\mathcal{A}$. Assertions of the form $A_k(a_m, [i, j])$ are encoded as:

$$
A_k(a_m, [i, j])^\flat = \begin{cases}
A_k(a_m) & \text{if } i = 0, j = 1 \\
\langle A \rangle(\mathsf{len}_{=(j-1)} \wedge [A]\langle \overline{A} \rangle(\mathsf{len}_{=|j-i|} \wedge A_k(a_m))) & \text{if } j > 1 \\
\langle \overline{A} \rangle(\mathsf{len}_{=|i|} \wedge [\overline{A}]\langle A \rangle(\mathsf{len}_{=|j-i|} \wedge A_k(a_m))) & \text{if } j \leq 1
\end{cases}
$$

Role assertions are slightly more complex. We can safely assume that if $S(a, b, [i, j]) \in \mathcal{A}$, then $\mathcal{A}$ also contains $S^-(b, a, [i, j])$. Since there are no binary predicates in the $\mathrm{FO}^1 \times \mathcal{HS}$ encoding we can only express the fact that a (resp., b) is in the domain (resp., range) of R. This is sufficient since *DL-Lite* logics do not have the expressive power to *qualify* those individuals whose existence is asserted. Furthermore, if R is rigid, then $R(a,b,[i,j]) \in \mathcal{A}$ implies that $R(a,b,[i',j'])$ holds for each $[i',j'] \in \mathbb{I}(\mathbb{Z})$. Thus, to count the number of individuals related via R to a at a certain interval $[i,j]$ we need to look at the *entire* ABox. Thus, for every $[i,j] \in \mathbb{I}(\mathbb{Z})$ and every R, we introduce the following *interval slice* $\mathcal{A}^R_{[i,j]}$ of $\mathcal{A}$ as

$$\mathcal{A}^R_{[i,j]} \;=\; \begin{cases} \{R(a,b) \mid R(a,b,[i',j']) \in \mathcal{A} \text{ for some } [i',j'] \in \mathbb{I}(\mathbb{Z})\}, & R \text{ rigid,} \\ \{R(a,b) \mid R(a,b,[i,j]) \in \mathcal{A}\}, & R \text{ flexible,} \end{cases}$$

and to count the number of R-successor of a at $[i,j]$ we use:

$$q^R_{a,[i,j]} = \max\{q \in Q_{\mathcal{K}} \mid R(a,b_1),\ldots,R(a,b_q) \in \mathcal{A}^R_{[i,j]}\}.$$

We can now define the translation $\mathcal{A}^\dagger$ of the ABox $\mathcal{A}$ as:

$$\bigwedge_{A_k(a,[i,j])\in\mathcal{A}} A_k(a,[i,j])^\flat \wedge \bigwedge_{\neg A_k(a,[i,j])\in\mathcal{A}} \neg A_k(a,[i,j])^\flat \wedge$$
$$\bigwedge_{R(a,b,[i,j])\in\mathcal{A}} E_{q^R_{a,[i,j]}} R(a,[i,j])^\flat \wedge \bigwedge_{\neg S(a,b,[i,j])\in\mathcal{A}, S(a,b,[i,j])\in\mathcal{A}^S_{[i,j]}} \bot.$$

where, by abuse of notation we denote with $E_{q^R_{a,[i,j]}} R(a,[i,j])^\flat$ the application of the $\cdot^\flat$ encoding — as presented above for ABox assertions — to ground predicates of the form $E_{q^R_{a,[i,j]}} R(a,[i,j])$. The above translation can be effectively computed since we need slices only for those intervals that are mentioned in the ABox. We finally define the encoding of $\mathcal{K}$ into an $\mathrm{FO}^1 \times \mathcal{HS}$ formula $\mathcal{K}^\dagger$ as the conjunction $\mathcal{T}^\dagger \wedge \mathcal{A}^\dagger$. We now establish the main result of this section, i.e., that $\mathcal{K}$ and $\mathcal{K}^\dagger$ are equi-satisfiable.

Theorem 1 *A $T_{\mathcal{HS}}$DL-Lite$^{\mathcal{N}}_{bool}$ knowledge base $\mathcal{K}$ is satisfiable iff the* $\mathrm{FO}^1 \times \mathcal{HS}$ *formula $\mathcal{K}^\dagger$ is satisfiable in* $[0,1]$.

Proof. If $\mathcal{K}$ is satisfiable over $\mathbb{Z}$, then it is trivial to see that $\mathcal{K}^\dagger$ is satisfied over the set $\mathbb{Z}$ in $[0,1]$. The difficult part is to show that the converse holds as well.

Suppose that for a given $\mathrm{FO}^1 \times \mathcal{HS}$ structure $\mathcal{M} = (\Delta^{\mathcal{M}}, \mathcal{M}(\cdot))$ based on $\mathbb{Z}$, it is the case that $\mathcal{M}([0,1]) \Vdash \mathcal{K}^\dagger$—notice that no assignment is involved as there are no free variables. We now show how to build a $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{N}}_{bool}$ interpretation $\mathcal{I}$ satisfying $\mathcal{K}$ following an *unfolding* technique already adopted in [8]. The domain $\Delta^{\mathcal{I}}$ is built inductively; we set $\Delta_0 = \{a^{\mathcal{M}} \mid a \in \mathsf{ob}(\mathcal{A})\}$ (assuming, w.l.g., that all $a^{\mathcal{M}}$ are distinct), and define:

$$\Delta^{\mathcal{I}} = \bigcup_{n\geq 0} \Delta_n.$$

Clearly, $a^{\mathcal{I}} = a^{\mathcal{M}}$ for every object. Intuitively, to build Δ_{n+1} from Δ_n, we add copies of elements $u \in \Delta^{\mathcal{M}} \setminus \Delta_0$ (i.e., we create elements that have the same properties as u). For an element $u \in \Delta^{\mathcal{I}}$, we denote by $\overrightarrow{u}$ that element in $\Delta^{\mathcal{M}}$ from which it has been copied, with $\overrightarrow{a} = a$, for any $a \in \mathsf{ob}(\mathcal{A})$. Therefore, for every interval $[i,j] \in \mathbb{I}(\mathbb{Z})$, and every concept name occurring in $\mathcal{K}$, we have that:

$$A^{\mathcal{I}([i,j])} = \{u \in \Delta^{\mathcal{I}} \mid \mathcal{M}([i,j]) \Vdash A^*(\overrightarrow{u})\}.$$

The interpretation of roles is defined along an inductive procedure as long as we construct the domain, i.e., $S^{\mathcal{I}([i,j])} = \bigcup_{n \geq 0} S_n^{[i,j]}$. Clearly, at each step n of the construction, $S_n^{[i,j]} \subseteq \Delta_n \times \Delta_n$ represents the (partial) interpretation of the role S at $[i,j]$. Given an individual $d \in \Delta^{\mathcal{M}}$, an interval $[i,j]$, and a role name R, we denote with $\rho_d^{R,[i,j]}$ the *required R-rank* of d at $[i,j]$ defined as:

$$\rho_d^{R,[i,j]} = \max(\{0\} \cup \{q \in Q_{\mathcal{K}} \mid \mathcal{M}([i,j]) \Vdash E_q R(d)\}).$$

Notice that $\rho_d^{R,[i,j]}$ is well defined: if $q = \rho_d^{R,[i,j]} > 0$ then $M([i,j]) \Vdash E_q R(d)$ and $M([i,j]) \Vdash E_{q'} R(d)$, for any $q' \in Q_{\mathcal{K}}$ with $q' < q$, while $M([i,j]) \nVdash E_{q'} R(d)$, for any $q' > q$. Similarly, the number of individuals that are actually R-related to some $u \in \Delta^{\mathcal{I}}$ at some interval $[i,j]$ and *some step n* of the construction can be computed by simply counting the number of distinct pairs $(u, u') \in S_n^{[i,j]}$, if $R = S$, or $(u', u) \in S_n^{[i,j]}$, if $R = S^-$. We denote this number as $\varsigma_{u,n}^{R,[i,j]}$ and call it the *actual R-rank*. Thus, the cardinalities of type ρ are based on the $\mathrm{FO}^1 \times \mathcal{HS}$ model $\mathcal{M}$, while cardinalities of type ς are based on the interpretation $\mathcal{I}$ at a certain step n of its construction. It is easy, now, to define the basis of the induction for building the role interpretation: for every role name S and every interval $[i,j]$, we put:

$$S_0^{[i,j]} = \{(a^{\mathcal{I}}, b^{\mathcal{I}}) \in \Delta_0 \times \Delta_0 \mid S(a,b) \in \mathcal{A}_{[i,j]}^S\}.$$

Since the interpretation of the roles starts with the information already present in $\mathcal{A}$, at the first step of the construction, the following holds: $\varsigma_{u,0}^{R,[i,j]} \leq \rho_{\overrightarrow{u}}^{R,[i,j]}$, for every $R \in \mathit{role}(\mathcal{K})$ and every interval $[i,j]$. The definition of $\mathcal{A}^\flat$, and, in particular, the third conjunct, guarantees this property. Now, suppose that we have defined Δ_n and $S_n^{[i,j]}$ for each interval $[i,j]$ up to a given step $n \geq 0$. If $\varsigma_{u,n}^{R,[i,j]} = \rho_{\overrightarrow{u}}^{R,[i,j]}$ for every $R \in \mathit{role}(\mathcal{K})$, the construction could be considered as finished. Suppose that for some role R and some individual u the number of its R-successors is less than the required R-rank. We can list all such *defects* at the step $n \geq 0$, relatively to each role $R \in \{S, S^-\}$ and the interval $[i,j]$ as:

$$\Lambda_{n,[i,j]}^R = \{u \in \Delta_n \setminus \Delta_{n-1} \mid \varsigma_{u,n}^{R,[i,j]} < \rho_{\overrightarrow{u}}^{R,[i,j]}\}.$$

We assume that $\Delta_{-1} = \emptyset$. At each step n of the construction, we build Δ_{n+1} from Δ_n by identifying the set of all defects, for each interval and role, and by fixing them. In doing so, we fix $S_{n+1}^{[i,j]}$ staring from $S_n^{[i,j]}$ by applying one of the following rules, depending on whether $u \in \Lambda_{n,[i,j]}^S$ or $u \in \Lambda_{n,[i,j]}^{S^-}$:

- Let $u \in \Lambda^{S}_{n,[i,j]}$ and $d = \overrightarrow{u}$. Thus, $q = \rho^{S,[i,j]}_{d} - \varsigma^{S,[i,j]}_{u,n} > 0$ counts the defects we need to repair. Then, $\mathcal{M}([i,j]) \Vdash E_{q'}S(d)$ for some $q' > q$. By $\mathcal{T}^{\flat}$ construction, we know that $\mathcal{M}([i,j]) \Vdash ES(d)$ and that $\mathcal{M}([i,j]) \Vdash ES^{-}(d')$ for some d'. We take q fresh copies $u_1, \ldots, u_q$ of d', so that $\overrightarrow{u}_i$ corresponds to d' for each $i = 1, \ldots, q$, and we add them to Δ_{n+1}. Also, we add $(u, u_1), \ldots, (u, u_q)$ to $S^{[i,j]}_{n+1}$. If S is rigid, then we add $S(u, u_1), \ldots, S(u, u_q)$ to $S^{[i',j']}_{n+1}$ for every $[i, j] \in \mathbb{I}(\mathbb{Z})$.
- The case $u \in \Gamma^{S-}_{n,[i,j]}$ is treated similarly.

It is immediate to see that after the application of the corresponding rule at the step n for the role S (resp., S^{-}), $u \in \Delta_n \setminus \Delta_{n-1}$ and interval $[i, j]$, $\rho^{S,[i,j]}_{\overrightarrow{u}} = \varsigma^{S,[i,j]}_{u,n+1}$ (resp., $\rho^{S^{-},[i,j]}_{\overrightarrow{u}} = \varsigma^{S^{-},[i,j]}_{u,n+1}$). It follows that, for every role $R \in role(\mathcal{K})$, $q \in Q_{\mathcal{K}}$, $[i, j] \in \mathbb{I}(\mathbb{Z})$, and $u \in \Delta^{\mathcal{I}}$, we have that:

$$\mathcal{M}([i,j]) \Vdash E_q R(\overrightarrow{u}) \text{ iff } u \in (\geq q\ R)^{\mathcal{I}([i,j])}. \tag{6}$$

It remains to be shown that

$$\mathcal{M}([i,j]) \Vdash C^{*}(\overrightarrow{u}) \text{ iff } u \in C^{\mathcal{I}([i,j])},$$

for each concept in $C \in \mathcal{K}$, every interval $[i, j]$, and every $u \in \Delta^{\mathcal{I}}$. We can do it by induction on the structure of C. If $C = \bot$ the case is trivial. If $C = A$, then the thesis follows by definition of $A^{\mathcal{I}([i,j])}$, while the case $C = \ \geq q\ R$ follows from (6). Finally, the inductive steps of $\neg$, $\sqcap$, and temporal operators are straightforward. Thus, $\mathcal{I} \Vdash \mathcal{T}$. Similarly, it is straightforward to prove that $\mathcal{I}$ satisfies every assertion in $\mathcal{A}$ over $[0, 1]$. ■

5.3 Embedding into Propositional $\mathcal{HS}$

We remark that every $\mathrm{FO}^1 \times \mathcal{HS}$-formula $\mathcal{K}^{\dagger}$ has the form:

$$\mathcal{K}^{\dagger} = \psi \wedge [G]\forall x \varphi(x) \wedge \bigwedge_{R \in role(\mathcal{K})} [G]\forall x (ER(x) \rightarrow \exists x (Einv(R)(x)))$$

where ψ is a ground formula. Thus, $\mathcal{K}^{\dagger}$ is an universal $\mathrm{FO}^1 \times \mathcal{HS}$-formula except for the last conjunct. We now show how to eliminate the use of the existential quantification in this last conjunct allowing us to translate $\mathcal{K}$ into a universally qualified $\mathrm{FO}^1 \times \mathcal{HS}$-formula. Consider the following universal formula, where p_R (resp., d_R) is a fresh propositional variable (resp., constant), for each $R \in role(\mathcal{K})$:

$$\mathcal{K}^{\ddagger} = \psi \wedge [G]\forall x \varphi(x) \wedge \tag{7}$$

$$\bigwedge_{R \in role(\mathcal{K})} \big[p_R \rightarrow (Einv(R)(d_{inv(R)}) \wedge [G]\forall x (ER(x) \rightarrow [G]p_R)) \big]. \tag{8}$$

Lemma 1 *Let $\mathcal{K}$ be a $T_{\mathcal{HS}}$DL-Lite$^{\mathcal{N}}_{bool}$ knowledge base. Then, $\mathcal{K}^{\dagger}$ is satisfiable iff $\mathcal{K}^{\ddagger}$ is satisfiable.*

Proof. ($\Rightarrow$) Let $\mathcal{M}$ be a model based on $\mathbb{Z}$ of $\mathcal{K}^{\dagger}$, i.e., $\mathcal{M}([0,1]) \Vdash \mathcal{K}^{\dagger}$. We first prove the existence of a model $\mathcal{M}'$ such that for every role name S in $\mathcal{K}$, $ES(x)$ and $ES^{-}(x)$ are either both empty or both non-empty for every interval $[i,j] \in \mathbb{I}(\mathbb{Z})$, and that $\mathcal{M}'([0,1]) \Vdash \mathcal{K}^{\dagger}$. To this end, suppose that for a role name S, some element $d \in \Delta^{\mathcal{M}}$, and some interval $[i,j] \in \mathbb{I}(\mathbb{Z})$, $\mathcal{M}([i,j]) \Vdash ES(d)$. By (4), we know that $\mathcal{M}([i,j]) \Vdash ES^{-}(d')$ for some other element $d' \in \Delta^{\mathcal{M}}$. We build a new model $\mathcal{M}'$ that extends both $\Delta^{\mathcal{M}}$ to $\Delta^{\mathcal{M}'}$ and the interpretation $p^{\mathcal{M}([i,j])}$ of each unary predicate $p(x)$ in $\mathcal{K}^{\dagger}$, for each $[i,j] \in \mathbb{I}(\mathbb{Z})$, as follows:

$$\begin{aligned}
\Delta^{\mathcal{M}'} &= \Delta^{\mathcal{M}} \cup (\{d, d'\} \times \mathbb{Z} \times \mathbb{Z}) \\
p^{\mathcal{M}'([i,j])} &= p^{\mathcal{M}([i,j])} \cup \{(d,k,k') \mid d \in p^{\mathcal{M}([i-k,j-k'])}, k,k' \in \mathbb{Z}\} \\
&\quad \cup \{(d',k,k') \mid d' \in p^{\mathcal{M}([i-k,j-k'])}, k,k' \in \mathbb{Z}\}.
\end{aligned}$$

To complete the model construction, we repeat the above steps for each role name in $\mathcal{K}$. To show that we obtained a new model for $\mathcal{K}^{\dagger}$ note that ψ only depends on the interpretation of constants and thus just on $\mathcal{M}$, the last conjunct is obviously true, while $[G]\forall x \varphi(x)$ is a unary predicate that must hold at every interval in $\mathbb{I}(\mathbb{Z})$ and thus it remains true in $\mathcal{M}'$—note that, this would be in general false for models over $\mathbb{N}$, e.g., consider as $\mathcal{K}^{\dagger}$ the formula $p(a) \wedge [G](\top \rightarrow [A]\neg p(x))$. Thus, if $\mathcal{K}^{\dagger}$ is satisfiable, then there exists a model, $\mathcal{M}^{\dagger}$, s.t. $\mathcal{M}^{\dagger}([0,1]) \Vdash \mathcal{K}^{\dagger}$ where, for every role name S in $\mathcal{K}$, the unary predicates ES and ES^{-} are either both empty or both non-empty for every interval $[i,j] \in \mathbb{I}(\mathbb{Z})$. Now, we want to build a model $\mathcal{M}^{\ddagger}$ that extends $\mathcal{M}^{\dagger}$ and satisfies $\mathcal{K}^{\ddagger}$. Consider a role name S. If ES and ES^{-} are both non-empty for every interval $[i,j] \in \mathbb{I}(\mathbb{Z})$, then we interpret p_S and p_{S^-} as true for every interval $[i,j] \in \mathbb{I}(\mathbb{Z})$, and d_S and d_{S^-} as any element from the interpretation of ES and ES^{-} in $\mathcal{M}^{\dagger}$ at the interval $[0,1]$. If, on the other hand, ES and ES^{-} are both empty for every interval $[i,j] \in \mathbb{I}(\mathbb{Z})$, we set p_S and p_{S^-} as false on $[0,1]$, and we interpret d_S and d_{S^-} as arbitrary domain elements. Now, under the assumption that (4) is satisfied in $\mathcal{M}^{\dagger}$, we have to prove that the last conjunct of $\mathcal{K}^{\ddagger}$ is satisfied in $\mathcal{M}^{\ddagger}$—the other conjuncts of $\mathcal{K}^{\ddagger}$ are identical to the corresponding ones in $\mathcal{K}^{\dagger}$, thus they are not affected by the extension from $\mathcal{M}^{\dagger}$ to $\mathcal{M}^{\ddagger}$. Consider a role name S, and assume that ES and ES^{-} are both non-empty for every interval $[i,j] \in \mathbb{I}(\mathbb{Z})$. By construction, p_S and p_{S^-} are true everywhere, proving that the second conjunct of (8) is satisfied, and since d_S (resp., d_{S^-}) is in the interpretation of ES (resp., ES^{-}) at $[0,1]$, the first conjunct is also satisfied. If ES and ES^{-} are both empty for every interval $[i,j] \in \mathbb{I}(\mathbb{Z})$, then, by construction, p_S and p_{S^-} are both false at $[0,1]$, proving that the first conjunct of (8) is satisfied; the second conjunct is satisfied in this case as ES and ES^{-} are both empty everywhere.

($\Leftarrow$) Conversely, we assume now that $\mathcal{K}^\ddagger$ is satisfied in some model $\mathcal{M}^\ddagger$ and we want to prove that so is the case for $\mathcal{K}^\dagger$. Notice that $\mathcal{M}^\ddagger$ satisfies all conjuncts of $\mathcal{K}^\dagger$ except, possibly, (4). If (4) is not satisfied in the model $\mathcal{M}^\ddagger$, then, there must be an interval $[i, j]$, a role R, and an element d of the domain such that $\mathcal{M}^\ddagger([i, j]) \Vdash ER(d)$ while $(Einv(R))^{\mathcal{M}^\ddagger([i,j])} = \emptyset$. By (8), this implies that p_R is true on every interval of the model, which, in turn, implies the existence of an element $d_{inv(R)} \in (Einv(R))^{\mathcal{M}^\ddagger([0,1])}$. We can therefore apply the same construction that we have applied before, to obtain an extension of $\mathcal{M}^\ddagger$ where, for every interval, the predicates ER and $Einv(R)$ are both non-empty. This model, say $\mathcal{M}^\dagger$, satisfies (4) and thus $\mathcal{M}^\dagger([0, 1]) \Vdash \mathcal{K}^\dagger$, as we wanted. ■

We conclude this part by observing that, after the above results, we have that the following theorem holds.

Theorem 2 *Let $\mathcal{K}$ be a $T_{\mathcal{HS}}$DL-Lite$^{\mathcal{N}}_{bool}$ knowledge base. Then, there exists an $\mathcal{HS}$-formula $\mathcal{K}^\ddagger$ such that $\mathcal{K}$ is satisfiable iff $\mathcal{K}^\ddagger$ is satisfiable.*

The above theorem is based on the straightforward observation that $\mathcal{K}^\ddagger$ is an $\mathrm{FO}^1 \times \mathcal{HS}$-formula with no existential quantifier, and it can be therefore considered as a purely propositional $\mathcal{HS}$-formula.

6 Complexity Results

Given any fragment $\mathcal{F}$ of $\mathcal{HS}$, the corresponding fragment $T_{\mathcal{F}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$ of $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$ is obtained by restricting the syntax of the language $T_{\mathcal{HS}}$*DL-Lite*$^{\mathcal{HN}}_{bool}$ to the temporal operators contained in $\mathcal{F}$. Theorem 2 gives us a powerful tool to identify decidable fragments of the latter: it is enough to pair *DL-Lite*$^{(\mathcal{HN})}_{bool}$ to decidable fragments of $\mathcal{HS}$. Those $\mathcal{HS}$-fragments, besides decidable, should be *suitable*, i.e., they should be able to capture both global formulas as in (3) and temporal ABoxes (see formula $A_k(a_m, [i, j])^\flat$). Both maximal fragments mentioned in Section 3 are suitable, which implies the following result.

Theorem 3 *The logics T_{MPNL}DL-Lite$^{(\mathcal{HN})}_{bool}$ and $T_{\mathsf{AB\overline{B}\overline{L}}}$DL-Lite$^{(\mathcal{HN})}_{bool}$ are maximal, expressively incomparable, and decidable, fragments of $T_{\mathcal{HS}}$DL-Lite$^{\mathcal{HN}}_{bool}$ interpreted over $\mathbb{Z}$. In both cases KB satisfiability is* ExpSpace*-complete.*

It is worth to point out that in the case of $T_{\mathsf{AB\overline{B}\overline{L}}}$*DL-Lite*$^{(\mathcal{HN})}_{bool}$, temporal Abox assertions in the past of the interval $[0, 1]$ requires a special and more involved treatment. For example, to say that $A(c)$ holds on the interval $[-2, -1]$ it is necessary to proceed as follows: *(i)* using a construction similar to [G], constraint the special proposition letter *Start* to hold only on $[0, 1]$ (simulate a *nominal*); *(ii)* then, use the formula $\langle\overline{L}\rangle(\langle A\rangle(\mathsf{len}_{=2} \wedge \langle A\rangle Starts) \wedge \langle A\rangle(\mathsf{len}_{=1} \wedge A(c)))$ to encode the temporal assertion.

When we interpret $T_{\mathcal{HS}}$*DL-Lite*$_{bool}^{\mathcal{HN}}$ over $\mathbb{N}$ we have to pay attention to the *suitability* condition since past operators are not allowed anymore. To capture ABox assertions over $\mathbb{N}$ we can still use length constraints but to replace the met-by ($\overline{\mathsf{A}}$) relation we need to extend the set of natural numbers to the set $\overline{\mathbb{N}} = \{-1\} \cup \mathbb{N}$ and interpret $\mathrm{FO}^1 \times \mathcal{HS}$-formulas on the interval $[-1, 0]$. Thus, assertions of the form $A_k(a_m, [i,j])$ are encoded in $\mathbb{N}$ as:

$$A_k(a_m, [i,j])^\partial \;=\; \begin{cases} \langle A\rangle(\mathsf{len}_{=\mathsf{j}} \wedge A_k(a_m)) & \text{if } i = 0,\\ \langle A\rangle(\mathsf{len}_{=\mathsf{i}} \wedge \langle A\rangle(\mathsf{len}_{=(\mathsf{j}-\mathsf{i})} \wedge A_k(a_m))) & \text{if } i > 0. \end{cases}$$

To capture the global modal operators (G) over $\mathbb{N}$ it is enough to interpret over $[-1, 0]$ the following formula using just the meet (A) temporal relation:

$$[G]\varphi \;=\; [A]\varphi \wedge [A][A]\varphi.$$

It is not difficult to see that the fragments of $\mathcal{HS}$ interpreted over $\mathbb{N}$ have exactly the same computational behaviour when they are interpreted over $\overline{\mathbb{N}}$. We can therefore conclude that the following theorem hold.

Theorem 4 *The fragments T_{MRPNL}DL-Lite and $T_{\mathsf{AB\overline{B}}}$DL-Lite, obtained by restricting the syntax of $T_{\mathcal{HS}}$DL-Lite$_{bool}^{\mathcal{HN}}$ to* MRPNL *(that is,* A *plus length constraints) and to* $\mathsf{AB\overline{B}}$*, respectively, are maximal decidable fragments of $T_{\mathcal{HS}}$DL-Lite$_{bool}^{\mathcal{HN}}$ interpreted over $\mathbb{N}$, and their satisfiability problem is* ExpSpace*-complete.*

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