Logic

Foundations of Propositional Logic

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### Knowledge bases

<table>
<thead>
<tr>
<th><strong>Inference engine</strong></th>
<th>← domain-independent algorithms</th>
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<tbody>
<tr>
<td><strong>Knowledge base</strong></td>
<td>← domain-specific content</td>
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- Knowledge base = set of *sentences* in a *formal* language = logical *theory*

- *Declarative* approach to building an intelligent agent:
  TELL it what it needs to know

- Then it can ASK itself what to do—answers should follow from the KB

- Agents can be viewed at the *knowledge level*
  i.e., what they know, regardless of how implemented

- Or at the *implementation level*
  i.e., data structures in KB and algorithms that manipulate them
Logic in general

- **Logics** are formal languages for representing information such that conclusions can be drawn.

- **Syntax** defines the sentences in the language.

- **Semantics** define the “meaning” of sentences; i.e., define *truth* of a sentence in a world.

- E.g., the language of arithmetic
  
  \[ x + 2 \geq y \text{ is a sentence; } x^2 + y > \text{ is not a sentence} \]
  \[ x + 2 \geq y \text{ is true iff the number } x + 2 \text{ is no less than the number } y \]
  \[ x + 2 \geq y \text{ is true in a world where } x = 7, \ y = 1 \]
  \[ x + 2 \geq y \text{ is false in a world where } x = 0, \ y = 6 \]
  \[ x + 2 \geq x + 1 \text{ is true in every world} \]
The one and only Logic?

- Logics of higher order
- Modal logics
  - epistemic
  - temporal and spatial
  - ...
- Description logic
- Non-monotonic logic
- Intuitionistic logic
- ...

But: There are “standard approaches”
  ⟷ propositional and predicate logic
Types of logic

- Logics are characterized by what they commit to as “primitives”
- Epistemological commitment: what states of knowledge?

<table>
<thead>
<tr>
<th>Language</th>
<th>Ontological Commitment (What exists in the world)</th>
<th>Epistemological Commitment (What an agent believes about facts)</th>
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<tbody>
<tr>
<td>Propositional logic</td>
<td>facts</td>
<td>true/false/unknown</td>
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<td>First-order logic</td>
<td>facts, objects, relations</td>
<td>true/false/unknown</td>
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<tr>
<td>Temporal logic</td>
<td>facts, objects, relations, times</td>
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<tr>
<td>Probability theory</td>
<td>facts</td>
<td>degree of belief 0…1</td>
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<tr>
<td>Fuzzy logic</td>
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Classical logics are based on the notion of TRUTH
Entailment – Logical Implication

Knowledge base $KB$ entails sentence $\alpha$ if and only if $\alpha$ is true in all worlds where $KB$ is true.

E.g., the KB containing “Manchester United won” and “Manchester City won” entails “Either Manchester United won or Manchester City won”
Models

• Logicians typically think in terms of *models*, which are formally *structured worlds* with respect to which truth can be evaluated.

• We say $m$ is a *model* of a sentence $\alpha$ if $\alpha$ is true in $m$.

• $M(\alpha)$ is the set of all models of $\alpha$.

• Then $KB \models \alpha$ if and only if $M(KB) \subseteq M(\alpha)$.

• E.g. $KB = \text{United won and City won}$
  
  $\alpha = \text{City won}$

  or

  $\alpha = \text{Manchester won}$

  or

  $\alpha = \text{either City or Manchester won}$
Inference – Deduction – Reasoning

\[ KB \vdash_i \alpha \]

- \( KB \vdash_i \alpha \) = sentence \( \alpha \) can be derived from \( KB \) by **procedure** \( i \)

- **Soundness**: \( i \) is sound if
  
  whenever \( KB \vdash_i \alpha \), it is also true that \( KB \models \alpha \)

- **Completeness**: \( i \) is complete if
  
  whenever \( KB \models \alpha \), it is also true that \( KB \vdash_i \alpha \)

- We will define a logic (first-order logic) which is expressive enough to say almost anything of interest, and for which there exists a sound and complete inference procedure.
Propositional Logics: Basic Ideas

Statements:

The elementary building blocks of propositional logic are *atomic statements* that cannot be decomposed any further: *propositions*. E.g.,

- “The block is red”
- “The proof of the pudding is in the eating”
- “It is raining”

and logical connectives “and”, “or”, “not”, by which we can build *propositional formulas*. 
Propositional Logics: Reasoning

We are interested in the questions:

- when is a statement \textbf{logically implied} by a set of statements, in symbols: \( \Theta \models \phi \)
- can we define \textbf{deduction} in such a way that deduction and entailment coincide?
Syntax of Propositional Logic

Countable alphabet $\Sigma$ of **atomic propositions**: $a, b, c, \ldots$

| $\phi, \psi$ | $\rightarrow$ | $a$ | **atomic formula** |
| $\perp$ | $\rightarrow$ | $false$ |
| $\top$ | $\rightarrow$ | $true$ |
| $\neg \phi$ | $\rightarrow$ | $negation$ |
| $\phi \land \psi$ | $\rightarrow$ | $conjunction$ |
| $\phi \lor \psi$ | $\rightarrow$ | $disjunction$ |
| $\phi \rightarrow \psi$ | $\rightarrow$ | $implication$ |
| $\phi \leftrightarrow \psi$ | $\rightarrow$ | $equivalence$ |

- **Atom**: atomic formula
- **Literal**: (negated) atomic formula
- **Clause**: disjunction of literals
Semantics: Intuition

• Atomic statements can be \textit{true} T or \textit{false} F.

• The truth value of formulas is determined by the truth values of the atoms (\textit{truth value assignment} or \textit{interpretation}).

Example: \((a \lor b) \land c\)

• If \(a\) and \(b\) are wrong and \(c\) is true, then the formula is not true.

• Then \textit{logical entailment} could be defined as follows:

• \(\phi\) is implied by \(\Theta\), if \(\phi\) is true in all “states of the world”, in which \(\Theta\) is true.
Semantics: Formally

A truth value assignment (or interpretation) of the atoms in $\Sigma$ is a function $\mathcal{I}$:

$$\mathcal{I}: \Sigma \rightarrow \{T, F\}.$$ 

Instead of $\mathcal{I}(a)$ we also write $a^{\mathcal{I}}$.

A formula $\phi$ is satisfied by an interpretation $\mathcal{I}$ ($\mathcal{I} \models \phi$) or is true under $\mathcal{I}$:

- $\mathcal{I} \models T$
- $\mathcal{I} \not\models \bot$
- $\mathcal{I} \models a$ \quad iff \quad $a^{\mathcal{I}} = T$
- $\mathcal{I} \models \neg \phi$ \quad iff \quad $\mathcal{I} \not\models \phi$
- $\mathcal{I} \models \phi \land \psi$ \quad iff \quad $\mathcal{I} \models \phi$ and $\mathcal{I} \models \psi$
- $\mathcal{I} \models \phi \lor \psi$ \quad iff \quad $\mathcal{I} \models \phi$ or $\mathcal{I} \models \psi$

$\mathcal{I} \models \phi \rightarrow \psi$ \quad iff \quad if $\mathcal{I} \models \phi$, then $\mathcal{I} \models \psi$

$\mathcal{I} \models \phi \leftrightarrow \psi$ \quad iff \quad $\mathcal{I} \models \phi$, if and only if $\mathcal{I} \models \psi$
Example

\[ I: \begin{cases} 
   a & \mapsto & T \\
   b & \mapsto & F \\
   c & \mapsto & F \\
   d & \mapsto & T \\
   \vdots & &
\end{cases} \]

\[ ((a \lor b) \leftrightarrow (c \lor d)) \land (\neg(a \land b) \lor (c \land \neg d)). \]
Exercise

- Find an interpretation and a formula such that the formula is true in that interpretation (or: the interpretation satisfies the formula).

- Find an interpretation and a formula such that the formula is not true in that interpretation (or: the interpretation does not satisfy the formula).

- Find a formula which can’t be true in any interpretation (or: no interpretation can satisfy the formula).
Satisfiability and Validity

An interpretation $\mathcal{I}$ is a **model** of $\phi$:

$$\mathcal{I} \models \phi$$

A formula $\phi$ is

- **satisfiable**, if there is some $\mathcal{I}$ that satisfies $\phi$,
- **unsatisfiable**, if $\phi$ is not satisfiable,
- **falsifiable**, if there is some $\mathcal{I}$ that does not satisfy $\phi$,
- **valid** (i.e., a **tautology**), if every $\mathcal{I}$ is a model of $\phi$.

Two formulas are **logically equivalent** ($\phi \equiv \psi$), if for all $\mathcal{I}$:

$$\mathcal{I} \models \phi \text{ iff } \mathcal{I} \models \psi$$
Exercise

Satisfiable, tautology?

$\((((a \land b) \leftrightarrow a) \rightarrow b)\)

$\(((\neg \phi \rightarrow \neg \psi) \rightarrow (\psi \rightarrow \phi))\)

$\((a \lor b \lor \neg c) \land (\neg a \lor \neg b \lor d) \land (\neg a \lor b \lor \neg d)\)$

Equivalent?

$\((\phi \lor (\psi \land \chi)) \equiv ((\phi \lor \psi) \land (\psi \land \chi))\)$

$\neg(\phi \lor \psi) \equiv \neg \phi \land \neg \psi$
Consequences

Proposition:

- $\phi$ is a tautology iff $\neg\phi$ is unsatisfiable
- $\phi$ is unsatisfiable iff $\neg\phi$ is a tautology.

Proposition: $\phi \equiv \psi$ iff $\phi \leftrightarrow \psi$ is a tautology.

Theorem: If $\phi$ and $\psi$ are equivalent, and $\chi'$ results from replacing $\phi$ in $\chi$ by $\psi$, then $\chi$ and $\chi'$ are equivalent.
Entailment

Extension of the entailment relationship to sets of formulas $\Theta$:

$$\mathcal{I} \models \Theta \iff \mathcal{I} \models \phi \text{ for all } \phi \in \Theta$$

Remember: we want the formula $\phi$ to be implied by a set $\Theta$, if $\phi$ is true in all models of $\Theta$ (symbolically, $\Theta \models \phi$):

$$\Theta \models \phi \iff \mathcal{I} \models \phi \text{ for all models } \mathcal{I} \text{ of } \Theta$$
Let $\alpha = A \lor B$ and $KB = (A \lor C) \land (B \lor \neg C)$

Is it the case that $KB \models \alpha$?

Check all possible models – $\alpha$ must be true wherever $KB$ is true

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(20/27)
Properties of Entailment

• $\Theta \cup \{\phi\} \models \psi$ iff $\Theta \models \phi \rightarrow \psi$

  (Deduction Theorem)

• $\Theta \cup \{\phi\} \models \neg\psi$ iff $\Theta \cup \{\psi\} \models \neg\phi$

  (Contraposition Theorem)

• $\Theta \cup \{\phi\}$ is unsatisfiable iff $\Theta \models \neg\phi$

  (Contradiction Theorem)
Equivalences (I)

Commutativity

\[
\phi \lor \psi \equiv \psi \lor \phi \\
\phi \land \psi \equiv \psi \land \phi \\
\phi \leftrightarrow \psi \equiv \psi \leftrightarrow \phi
\]

Associativity

\[
(\phi \lor \psi) \lor \chi \equiv \phi \lor (\psi \lor \chi) \\
(\phi \land \psi) \land \chi \equiv \phi \land (\psi \land \chi)
\]

Idempotence

\[
\phi \lor \phi \equiv \phi \\
\phi \land \phi \equiv \phi
\]

Absorption

\[
\phi \lor (\phi \land \psi) \equiv \phi \\
\phi \land (\phi \lor \psi) \equiv \phi
\]

Distributivity

\[
\phi \land (\psi \lor \chi) \equiv (\phi \land \psi) \lor (\phi \land \chi) \\
\phi \lor (\psi \land \chi) \equiv (\phi \lor \psi) \land (\phi \lor \chi)
\]
Equivalences (II)

**Tautology**
\[ \phi \lor T \equiv T \]

**Unsatisfiability**
\[ \phi \land \bot \equiv \bot \]

**Negation**
\[ \phi \lor \neg \phi \equiv T \]
\[ \phi \land \neg \phi \equiv \bot \]

**Neutrality**
\[ \phi \land T \equiv \phi \]
\[ \phi \lor \bot \equiv \phi \]

**Double Negation**
\[ \neg \neg \phi \equiv \phi \]

**De Morgan**
\[ \neg (\phi \lor \psi) \equiv \neg \phi \land \neg \psi \]
\[ \neg (\phi \land \psi) \equiv \neg \phi \lor \neg \psi \]

**Implication**
\[ \phi \rightarrow \psi \equiv \neg \phi \lor \psi \]
Normal Forms

Other approaches to inference use syntactic operations on sentences, often expressed in standardized forms

**Conjunctive Normal Form (CNF)**

- **conjunction of disjunctions of literals:** \( \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} l_{i,j} \)

  E.g., \((A \lor \neg B) \land (B \lor \neg C \lor \neg D)\)

**Disjunctive Normal Form (DNF)**

- **disjunction of conjunctions of literals:** \( \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} l_{i,j} \)

  E.g., \((A \land B) \lor (A \land \neg C) \lor (A \land \neg D) \lor (\neg B \land \neg C) \lor (\neg B \land \neg D)\)
Normal Forms, cont.

Horn Form (restricted)

*conjunction of Horn clauses* (clauses with \( \leq 1 \) positive literal)

E.g., \((A \lor \neg B) \land (B \lor \neg C \lor \neg D)\)

Often written as set of implications:

\(B \Rightarrow A \text{ and } (C \land D) \Rightarrow B\)

**Theorem** For every formula, there exists an equivalent formula in CNF and one in DNF.
Why Normal Forms?

• We can transform propositional formulas, in particular, we can construct their CNF and DNF.

• DNF tells us something as to whether a formula is satisfiable. If all disjuncts contain $\bot$ or complementary literals, then no model exists. Otherwise, the formula is satisfiable.

• CNF tells us something as to whether a formula is a tautology. If all clauses (= conjuncts) contain $\top$ or complementary literals, then the formula is a tautology. Otherwise, the formula is falsifiable.

But:

• the transformation into DNF or CNF is expensive (in time/space)

• it is only possible for finite sets of formulas
Summary: important notions

- Syntax: formula, atomic formula, literal, clause
- Semantics: truth value, assignment, interpretation
- Formula satisfied by an interpretation
- Logical implication, entailment
- Satisfiability, validity, tautology, logical equivalence
- Deduction theorem, Contraposition Theorem
- Conjunctive normal form, Disjunctive Normal form, Horn form