CHAPTER 7

FUNCTIONS

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SECTION 7.1

Functions Defined on General Sets

Functions Defined on General Sets

The following is the definition of a **function** that includes additional terminology associated.

• Definition

A **function** f **from a set** X **to a set** Y, denoted $f: X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically,

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

Functions Acting on Sets

Functions Acting on Sets

Given a function from a set X to a set Y, you can consider the set of images in Y of all the elements in a subset of X and the set of inverse images in X of all the elements in a subset of Y.

• Definition

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

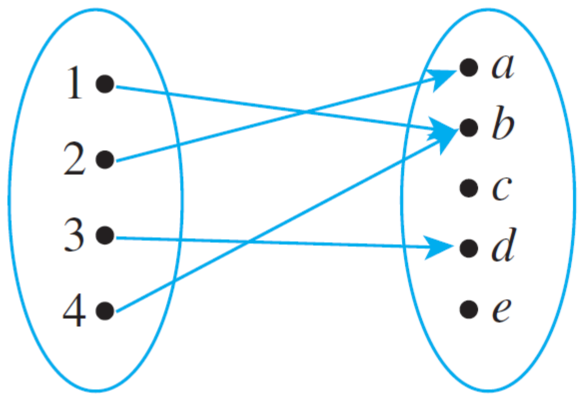
and

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

Example 13 – *The Action of a Function on Subsets of a Set*

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$, and define $F : X \to Y$ by the following arrow diagram:



Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$.

Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

Example 13 – *Solution*

$$F(A) = \{b\}$$

$$F(X) = \{a, b, d\}$$

$$F^{-1}(C) = \{1, 2, 4\}$$

$$F^{-1}(D) = \emptyset$$

SECTION 7.2

One-to-One and Onto, Inverse Functions

One-to-One and Onto, Inverse Functions

In this section we discuss two important properties that functions may satisfy: the property of being *one-to-one* and the property of being *onto*.

Functions that satisfy both properties are called *one-to-one correspondences* or *one-to-one and onto functions*.

When a function is a one-to-one correspondence, the elements of its domain and co-domain match up perfectly, and we can define an *inverse function* from the co-domain to the domain that "undoes" the action of the function.

One-to-One Functions

One-to-One Functions

We have noted earlier that a function may send several elements of its domain to the same element of its co-domain.

In terms of arrow diagrams, this means that two or more arrows that start in the domain can point to the same element in the co-domain.

If no two arrows that start in the domain point to the same element of the co-domain then the function is called *one-to-one* or *injective*.

One-to-One Functions

For a one-to-one function, each element of the range is the image of at most one element of the domain.

• Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

To obtain a precise statement of what it means for a function *not* to be one-to-one, take the negation of one of the equivalent versions of the definition above.

One-to-One Functions

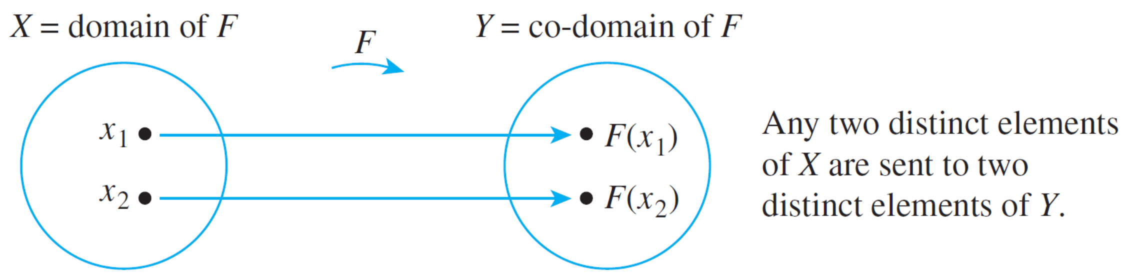
Thus:

A function $F: X \to Y$ is *not* one-to-one	$\Leftrightarrow$	$\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

That is, if elements x_1 and x_2 exist that have the same function value but are not equal, then F is not one-to-one.

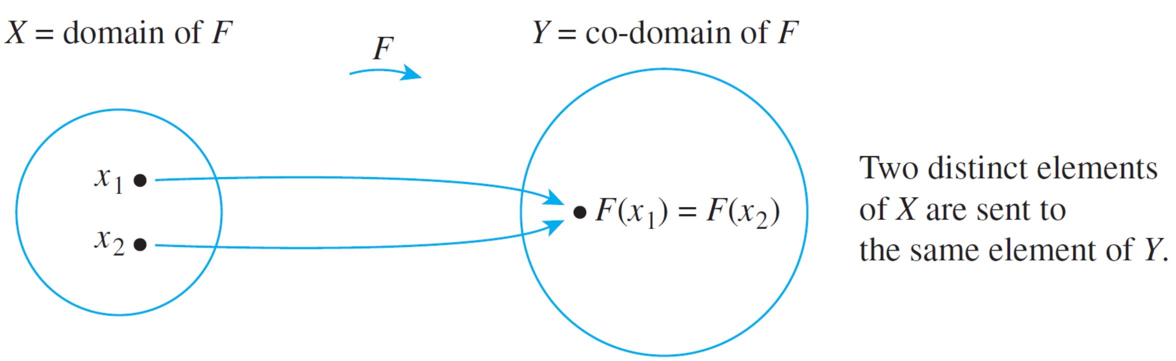
One-to-One Functions

This is illustrated in Figure 7.2.1



A One-to-One Function Separates Points

Figure 7.2.1 (a)



A Function That Is Not One-to-One Collapses Points Together

Figure 7.2.1 (b)

One-to-One Functions on Infinite Sets

One-to-One Functions on Infinite Sets

Now suppose f is a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Thus, to prove f is one-to-one, you will generally use the method of direct proof:

suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$

and **show** that $x_1 = x_2$.

One-to-One Functions on Infinite Sets

To show that f is *not* one-to-one, you will ordinarily

> **find** elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Example – *Proving or Disproving That Functions Are One-to-One*

Define f: $\mathbf{R} \to \mathbf{R}$ and g: $\mathbf{Z} \to \mathbf{Z}$ by the rules.

$$f(x) = 4x - 1 \quad \text{for all} \quad x \in \mathbf{R}$$

and

$$g(n) = n^2 \quad \text{for all} \quad n \in \mathbf{Z}.$$

a. Is f one-to-one? Prove or give a counterexample.

b. Is g one-to-one? Prove or give a counterexample.

Example 2 – *Solution*

It is usually best to start by taking a positive approach to answering questions like these. Try to prove the given functions are one-to-one and see whether you run into difficulty.

If you finish without running into any problems, then you have a proof. If you do encounter a problem, then analyzing the problem may lead you to discover a counterexample.

a. The function *f*: **R** → **R** is defined by the rule

$$f(x) = 4x - 1 \quad \text{for all real numbers } x.$$

Example 2 – *Solution*

cont'd

To prove that *f* is one-to-one, you need to prove that

$$\forall \text{ real numbers } x_1 \text{ and } x_2, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Substituting the definition of *f* into the outline of a direct proof, you

suppose x_1 and x_2 are any real numbers such that $4x_1 - 1 = 4x_2 - 1$,

and **show** that $x_1 = x_2$.

Example 2 – *Solution*

cont'd

Can you reach what is to be shown from the supposition?

Of course. Just add 1 to both sides of the equation in the supposition and then divide both sides by 4.

This discussion is summarized in the following formal answer.

Answer to (a):

If the function $f: \mathbf{R} \to \mathbf{R}$ is defined by the rule $f(x) = 4x - 1$, for all real numbers x, then f is one-to-one.

Example 2 – *Solution*

cont'd

Proof:

Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$.

[We must show that $x_1 = x_2$.]

By definition of *f*,

$$4x_1 - 1 = 4x_2 - 1.$$

Adding 1 to both sides gives

$$4x_1 = 4x_2,$$

and dividing both sides by 4 gives

$$x_1 = x_2,$$

which is what was to be shown.

Example 2 – *Solution*

cont'd

b. The function g: **Z** → **Z** is defined by the rule

$$\boxed{g(n) = n^2 \quad \text{for all integers } n.}$$

As above, you start as though you were going to prove that g is one-to-one.

Substituting the definition of g into the outline of a direct proof, you

suppose n_1 and n_2 are integers such that $n_1^2 = n_2^2$,

and **try to show** that $n_1 = n_2$.

Example 2 – *Solution*

cont' d

Can you reach what is to be shown from the supposition? No! It is quite possible for two numbers to have the same squares and yet be different.

For example, $2^2 = (-2)^2$ but $2 \neq -2$.

Thus, in trying to prove that g is one-to-one, you run into difficulty.

But analyzing this difficulty leads to the discovery of a counterexample, which shows that g is not one-to-one.

Example 2 – *Solution*

cont'd

This discussion is summarized as follows:

> **Answer to (b):**
>
> If the function $g: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $g(n) = n^2$, for all $n \in \mathbf{Z}$, then g is not one-to-one.

Counterexample:

Let $n_1 = 2$ and $n_2 = 2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \quad \text{and also}$$

$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $$g(n_1) = g(n_2) \quad \text{but} \quad n_1 \neq n_2,$$

and so g is not one-to-one.

Onto Functions

Onto Functions

We noted that there may be elements of the co-domain of a function that are not the image of any element in the domain.

When a function is onto, its range is equal to its co-domain.

Such functions are called ***onto*** or ***surjective***.

• Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Onto Functions

To obtain a precise statement of what it means for a function *not* to be onto, take the negation of the definition of onto:

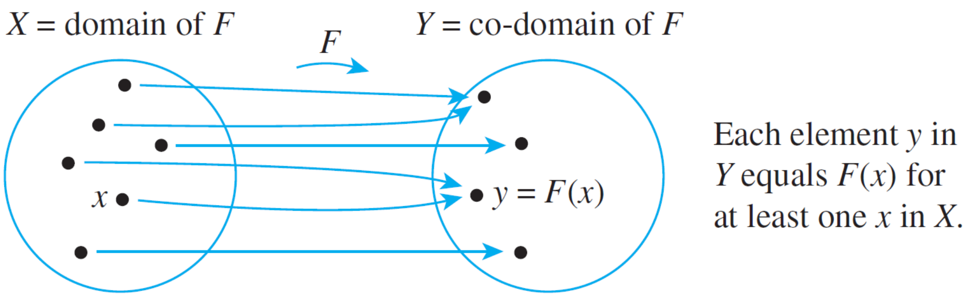
$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X,\ F(x) \neq y.$$

That is, there is some element in *Y* that is *not* the image of *any* element in *X*.

In terms of arrow diagrams, a function is onto if each element of the co-domain has an arrow pointing to it from some element of the domain. A function is not onto if at least one element in its co-domain does not have an arrow pointing to it.

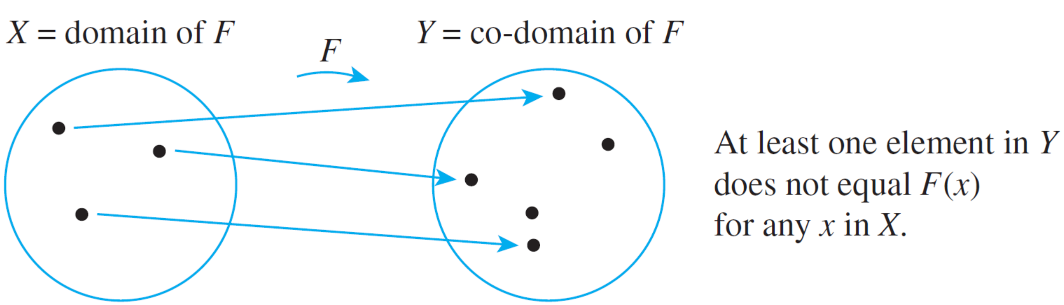
Onto Functions

This is illustrated in Figure 7.2.3.



A Function That Is Onto

Figure 7.2.3 (a)



A Function That Is Not Onto

Figure 7.2.3 (b)

Onto Functions on Infinite Sets

Onto Functions on Infinite Sets

Now suppose F is a function from a set X to a set Y, and suppose Y is infinite. By definition, F is onto if, and only if, the following universal statement is true:

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Thus to prove F is onto, you will ordinarily use the method of generalizing from the generic particular:

suppose that y is any element of Y

and **show** that there is an element x of X with $F(x) = y$.

To prove F is *not* onto, you will usually

find an element y of Y such that $y \neq F(x)$ for *any* x in X.

Example 5 – *Proving or Disproving That Functions Are Onto*

Define f: $\mathbf{R} \to \mathbf{R}$ and h: $\mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R}$$

And

$$h(n) = 4n - 1 \quad \text{for all } n \in \mathbf{Z}.$$

a. Is f onto? Prove or give a counterexample.

b. Is h onto? Prove or give a counterexample.

Example 5 – *Solution*

a. The best approach is to start trying to prove that *f* is onto and be alert for difficulties that might indicate that it is not. Now *f*: **R** → **R** is the function defined by the rule

$$\boxed{f(x) = 4x - 1 \quad \text{for all real numbers } x.}$$

To prove that *f* is onto, you must prove

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

Example 5 – *Solution*

cont'd

Substituting the definition of f into the outline of a proof by the method of generalizing from the generic particular, you

suppose y is a real number

and **show** that there exists a real number x such that $y = 4x - 1$.

Scratch Work: If such a real number x exists, then

$$4x - 1 = y$$

$$4x = y + 1 \quad \text{by adding 1 to both sides}$$

$$x = \frac{y+1}{4} \quad \text{by dividing both sides by 4.}$$

Example 5 – *Solution*

cont'd

Thus *if* such a number x exists, it must equal $(y + 1)/4$. Does such a number exist? Yes.

To show this, let $x = (y + 1)/4$, and then made sure that

(1) x is a real number and that

(2) f really does send x to y.

The following formal answer summarizes this process.

Answer to (a):

If $f: \mathbf{R} \to \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1$ for all real numbers x, then f is onto.

Example 5 – *Solution*

cont' d

Proof:

Let $y \in \mathbf{R}$. *[We must show that* $\exists\, x$ *in* **R** *such that* $f(x) = y$.*]*
Let $x = (y + 1)/4$.

Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$
\begin{aligned}
f(x) &= f\left(\frac{y+1}{4}\right) && \text{by substitution} \\
&= 4 \cdot \left(\frac{y+1}{4}\right) - 1 && \text{by definition of } f \\
&= (y+1) - 1 = y && \text{by basic algebra.}
\end{aligned}
$$

[This is what was to be shown.]

Example 5 – *Solution*

cont'd

b. The function h: $\mathbf{Z} \to \mathbf{Z}$ is defined by the rule

$$\boxed{h(n) = 4n - 1 \quad \text{for all integers } n.}$$

To prove that h is onto, it would be necessary to prove that

$$\forall \text{ integers } m, \exists \text{ an integer } n \text{ such that } h(n) = m.$$

Substituting the definition of h into the outline of a proof by the method of generalizing from the generic particular, you

suppose m is any integer

and **try to show** that there is an integer n with $4n - 1 = m$.

Example 5 – *Solution*

cont'd

Can you reach what is to be shown from the supposition? No! If $4n - 1 = m$, then

$$n = \frac{m+1}{4} \qquad \text{by adding 1 and dividing by 4.}$$

But n must be an integer. And when, for example, $m = 0$, then

$$n = \frac{0+1}{4} = \frac{1}{4},$$

which is *not* an integer.

Thus, in trying to prove that h is onto, you run into difficulty, and this difficulty reveals a counterexample that shows h is not onto.

One-to-One Correspondences

One-to-One Correspondences

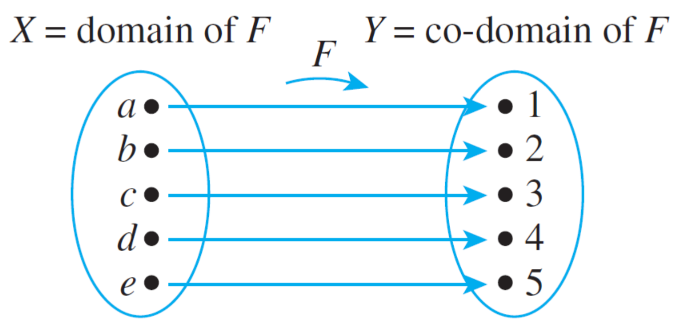
Consider a function $F: X \to Y$ that is both one-to-one and onto.

- Given any element x in X, there is a unique corresponding element $y = F(x)$ in Y (since F is a function).
- Also given any element y in Y, there is an element x in X such that $F(x) = y$ (since F is onto), and
- There is only one such x (since F is one-to-one).

One-to-One Correspondences

Thus, a function that is one-to-one and onto sets up a pairing between the elements of X and the elements of Y that matches each element of X with exactly one element of Y and each element of Y with exactly one element of X.

Such a pairing is called a ***one-to-one correspondence*** or ***bijection*** and is illustrated by the arrow diagram in Figure 7.2.5.



An Arrow Diagram for a One-to-One Correspondence

Figure 7.2.5

One-to-One Correspondences

One-to-one correspondences are often used as aids to counting.

The pairing of Figure 7.2.5, for example, shows that there are five elements in the set *X*.

• Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.

Inverse Functions

Inverse Functions

If F is a one-to-one correspondence from a set X to a set Y, then there is a function from Y to X that "undoes" the actions of F; that is, it sends each element of Y back to the element of X that it came from. This function is called the ***inverse function*** for F.

Theorem 7.2.2

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:

Given any element y in Y,

$$\boldsymbol{F^{-1}(y)} = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

Inverse Functions

The proof of Theorem 7.2.2 follows immediately from the definition of one-to-one and onto.

Given an element y in Y, there is an element x in X with $F(x) = y$ because F is onto; x is unique because F is one-to-one.

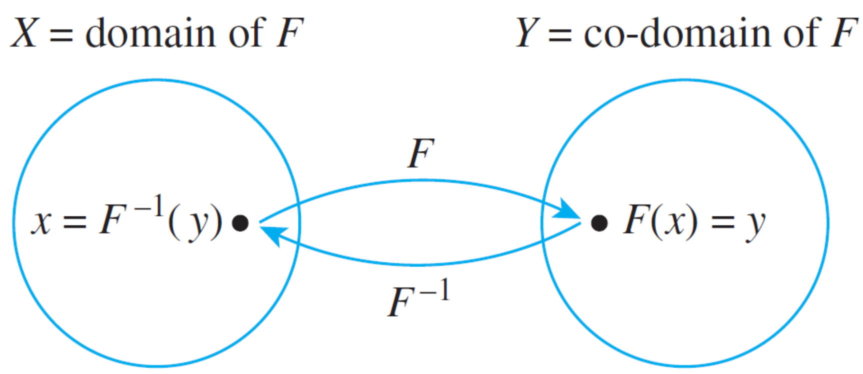
• Definition

The function F^{-1} of Theorem 7.2.2 is called the **inverse function** for F.

Note that according to this definition, the logarithmic function with base $b > 0$ is the inverse of the exponential function with base b.

Inverse Functions

The diagram that follows illustrates the fact that an inverse function sends each element back to where it came from.



Inverse Functions

Theorem 7.2.3

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

Proof:

F^{-1} is one-to-one: Let y_1 and y_2 be elements of Y such that $F^{-1}(y_1) = F^{-1}(y_2)$. *[We must show that $y_1 = y_2$.]* Let $x = F^{-1}(y_1) = F^{-1}(y_2)$. Then $x \in X$, and by definition of F^{-1},

$F(x) = y_1$ since $x = F^{-1}(y_1)$, and $F(x) = y_2$ since $x = F^{-1}(y_2)$. Consequently, $y_1 = y_2$ since each is equal to $F(x)$.

F^{-1} is onto: Let $x \in X$. *[We must show that there exists an element y in Y such that $F^{-1}(y) = x$.]* Let $y = F(x)$. Then $y \in Y$, and by definition of F^{-1}, $F^{-1}(y) = x$.