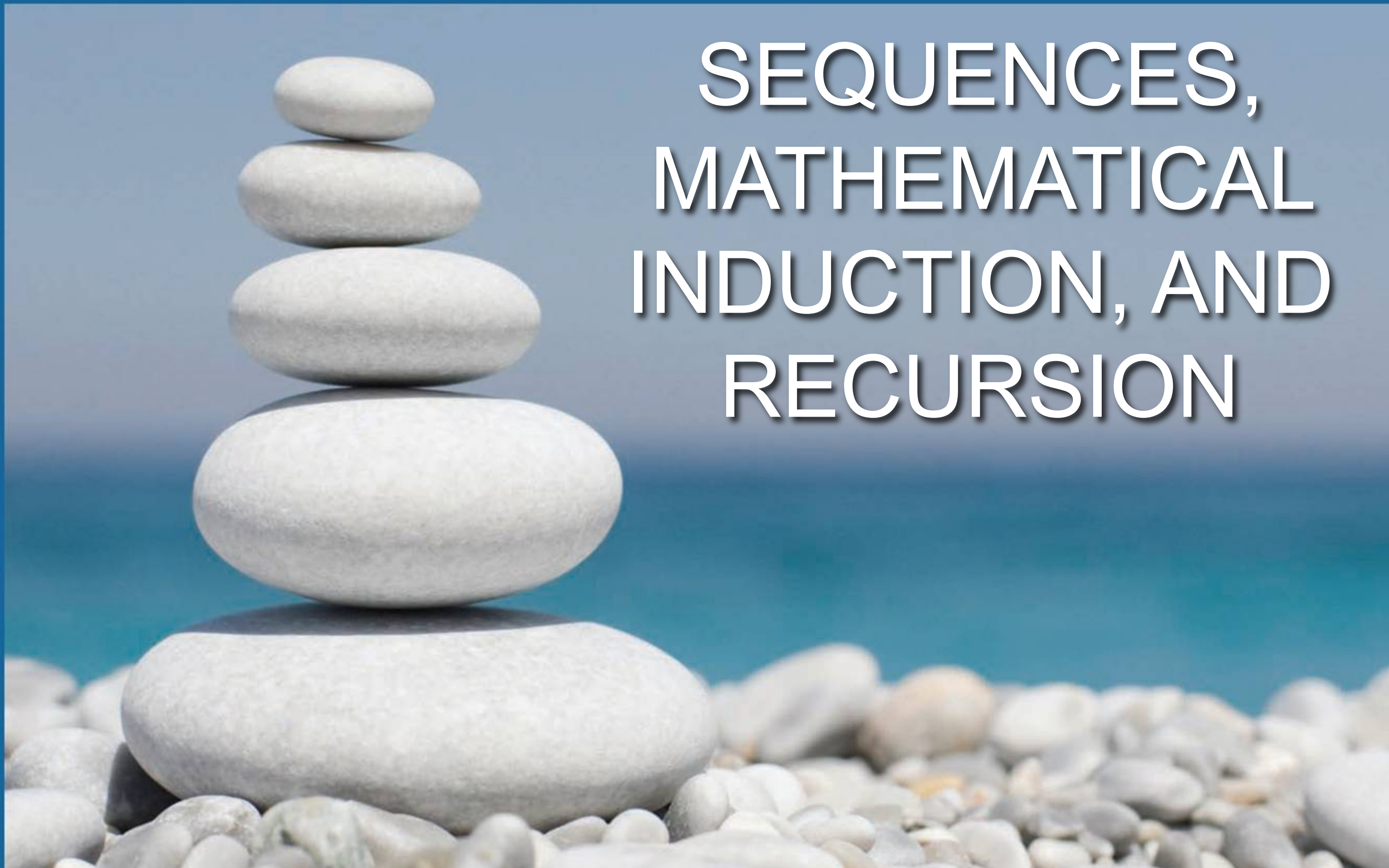
CHAPTER 5

SEQUENCES, MATHEMATICAL INDUCTION, AND RECURSION



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SECTION 5.2

Mathematical Induction I

Mathematical Induction I

Mathematical induction is one of the more recently developed techniques of proof in the history of mathematics.

It is used to check conjectures about the outcomes of processes that occur repeatedly and according to definite patterns.

In general, mathematical induction is a method for proving that a property defined for integers n is true for all values of n that are greater than or equal to some initial integer.

Mathematical Induction I

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.

2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true.

The validity of proof by mathematical induction is generally taken as an axiom. That is why it is referred to as the *principle* of mathematical induction rather than as a theorem.

Mathematical Induction I

Proving a statement by mathematical induction is a two-step process. The first step is called the *basis step*, and the second step is called the *inductive step*.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.***]*

Then

show that $P(k+1)$ is true.

Mathematical Induction I

The following example shows how to use mathematical induction to prove a formula for the sum of the first *n* integers.

Example 1 – *Sum of the First n Integers*

Use mathematical induction to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1.$$

Solution:

To construct a proof by induction, you must first identify the property *P*(*n*). In this case, *P*(*n*) is the equation

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}.} \qquad \leftarrow \text{the property } (P(n))$$

[To see that P(n) is a sentence, note that its subject is "the sum of the integers from 1 to n" and its verb is "equals."]

Example 1 – *Solution*

cont'd

In the basis step of the proof, you must show that the property is true for n = 1, or, in other words that P(1) is true.

Now P(1) is obtained by substituting 1 in place of n in P(n). The left-hand side of P(1) is the sum of all the successive integers starting at 1 and ending at 1. This is just 1. Thus P(1) is

$$\boxed{1 = \frac{1(1+1)}{2}.} \qquad \leftarrow \text{basis } (P(1))$$

Example 1 – *Solution*

cont'd

Of course, this equation is true because the right-hand side is

$$\frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1,$$

which equals the left-hand side.

In the inductive step, you assume that $P(k)$ is true, for a particular but arbitrarily chosen integer k with $k \geq 1$. *[This assumption is the inductive hypothesis.]*

Example 1 – *Solution*

cont' d

You must then show that $P(k + 1)$ is true. What are $P(k)$ and $P(k + 1)$? $P(k)$ is obtained by substituting k for every n in $P(n)$.

Thus $P(k)$ is

$$\boxed{1 + 2 + \cdots + k = \frac{k(k+1)}{2}.} \qquad \leftarrow \text{inductive hypothesis} \quad (P(k))$$

Similarly, $P(k + 1)$ is obtained by substituting the quantity $(k + 1)$ for every n that appears in $P(n)$.

Example 1 – *Solution*

cont'd

Thus $P(k + 1)$ is

$$1 + 2 + \cdots + (k + 1) = \frac{(k + 1)((k + 1) + 1)}{2},$$

or, equivalently,

$$\boxed{1 + 2 + \cdots + (k + 1) = \frac{(k + 1)(k + 2)}{2}.} \quad \leftarrow \text{ to show } (P(k+1))$$

Example 1 – *Solution*

cont'd

Now the inductive hypothesis is the supposition that $P(k)$ is true. How can this supposition be used to show that $P(k + 1)$ is true? $P(k + 1)$ is an equation, and the truth of an equation can be shown in a variety of ways.

One of the most straightforward is to use the inductive hypothesis along with algebra and other known facts to transform separately the left-hand and right-hand sides until you see that they are the same.

Example 1 – *Solution*

cont'd

In this case, the left-hand side of $P(k + 1)$ is

$$1 + 2 + \cdots + (k + 1),$$

which equals

$$(1 + 2 + \cdots + k) + (k + 1)$$

The next-to-last term is k because the terms are successive integers and the last term is $k + 1$.

But by substitution from the inductive hypothesis,

$$(1 + 2 + \cdots + k) + (k + 1)$$

$$= \frac{k(k+1)}{2} + (k + 1)$$

since the inductive hypothesis says that $1 + 2 + \cdots + k = \dfrac{k(k+1)}{2}$

Example 1 – *Solution*

cont' d

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

by multiplying the numerator and denominator of the second term by 2 to obtain a common denominator

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

by multiplying out the two numerators

$$= \frac{k^2+3k+2}{2}$$

by adding fractions with the same denominator and combining like terms.

Example 1 – *Solution*

cont'd

So the left-hand side of $P(k + 1)$ is $\dfrac{k^2 + 3k + 2}{2}$.

Now the right-hand side of $P(k + 1)$ is $\dfrac{(k+1)(k+2)}{2} = \dfrac{k^2 + 3k + 2}{2}$ by multiplying out the numerator.

Thus the two sides of $P(k + 1)$ are equal to each other, and so the equation $P(k + 1)$ is true.

Mathematical Induction I

In a **geometric sequence,** each term is obtained from the preceding one by multiplying by a constant factor.

If the first term is 1 and the constant factor is r, then the sequence is $1, r, r^2, r^3, \ldots, r^n, \ldots$.

The sum of the first n terms of this sequence is given by the formula

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

for all integers $n \geq 0$ and real numbers r not equal to 1.

Mathematical Induction I

The expanded form of the formula is

$$r^0 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1},$$

and because $r^0 = 1$ and $r^1 = r$, the formula for $n \geq 1$ can be rewritten as

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Example 3 – *Sum of a Geometric Sequence*

Prove that $\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$, for all integers $n \geq 0$ and all real numbers r except 1.

Solution:

In this example the property $P(n)$ is again an equation, although in this case it contains a real variable r:

$$\boxed{\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.} \qquad \leftarrow \text{the property } (P(n))$$

Example 3 – *Solution*

cont'd

Because r can be any real number other than 1, the proof begins by supposing that r is a particular but arbitrarily chosen real number not equal to 1.

Then the proof continues by mathematical induction on n, starting with $n = 0$.

In the basis step, you must show that $P(0)$ is true; that is, you show the property is true for $n = 0$.

Example 3 – *Solution*

cont'd

So you substitute 0 for each *n* in *P*(*n*):

$$\boxed{\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1}.} \qquad \leftarrow \text{basis } (P(0))$$

In the inductive step, you suppose *k* is any integer with *k* ≥ 0 for which *P*(*k*) is true; that is, you suppose the property is true for *n* = *k*.

Example 3 – *Solution*

cont'd

So you substitute *k* for each *n* in *P*(*n*):

$$\boxed{\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1}.} \qquad \leftarrow \text{inductive hypothesis } (P(k))$$

Then you show that *P*(*k* + 1) is true; that is, you show the property is true for *n* = *k* + 1.

So you substitute *k* + 1 for each *n* in *P*(*n*):

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1},$$

Example 3 – *Solution*

cont'd

Or, equivalently,

$$\boxed{\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}.} \qquad \leftarrow \text{to show } (P(k+1))$$

In the inductive step for this proof we use another common technique for showing that an equation is true:

> We start with the left-hand side and transform it step-by-step into the right-hand side using the inductive hypothesis together with algebra and other known facts.

Example 3 – *Solution*

cont'd

Theorem 5.2.3 Sum of a Geometric Sequence

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Proof (by mathematical induction):

Suppose *r* is a particular but arbitrarily chosen real number that is not equal to 1, and let the property *P*(*n*) be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{i+1} - 1}{r - 1} \qquad \leftarrow P(n)$$

We must show that *P*(*n*) is true for all integers $n \geq 0$. We do this by mathematical induction on *n*.

Example 3 – *Solution*

cont'd

Show that *P*(0) is true:

To establish *P*(0), we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} \qquad \leftarrow P(0)$$

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence *P*(0) is true.

Example 3 – *Solution*

cont' d

Show that for all integers *k* ≥ 0, if *P*(*k*) is true then *P*(*k* + 1) is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 0. *That is:]*

Let *k* be any integer with *k* ≥ 0, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

*[We must show that P(*k* + 1) is true. That is:]* We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1}.$$

Example 3 – *Solution*

cont’d

Or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}. \qquad \leftarrow P(k+1)$$

[We will show that the left-hand side of P(*k* + 1) *equals the right-hand side.]* The left-hand side of *P*(*k* + 1) is

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1}$$

by writing the $(k+1)$st term separately from the first k terms

$$= \frac{r^{k+1}-1}{r-1} + r^{k+1}$$

by substitution from the inductive hypothesis

Example 3 – *Solution*

cont' d

$$= \frac{r^{k+1} - 1}{r - 1} + \frac{r^{k+1}(r - 1)}{r - 1}$$

by multiplying the numerator and denominator of the second term by $(r - 1)$ to obtain a common denominator

$$= \frac{(r^{k+1} - 1) + r^{k+1}(r - 1)}{r - 1}$$

by adding fractions

$$= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r - 1}$$

by multiplying out and using the fact that $r^{k+1} \cdot r = r^{k+1} \cdot r^1 = r^{k+2}$

$$= \frac{r^{k+2} - 1}{r - 1}$$

by canceling the r^{k+1}'s.

which is the right-hand side of $P(k + 1)$ *[as was to be shown.]*

[Since we have proved the basis step and the inductive step, we conclude that the theorem is true.]