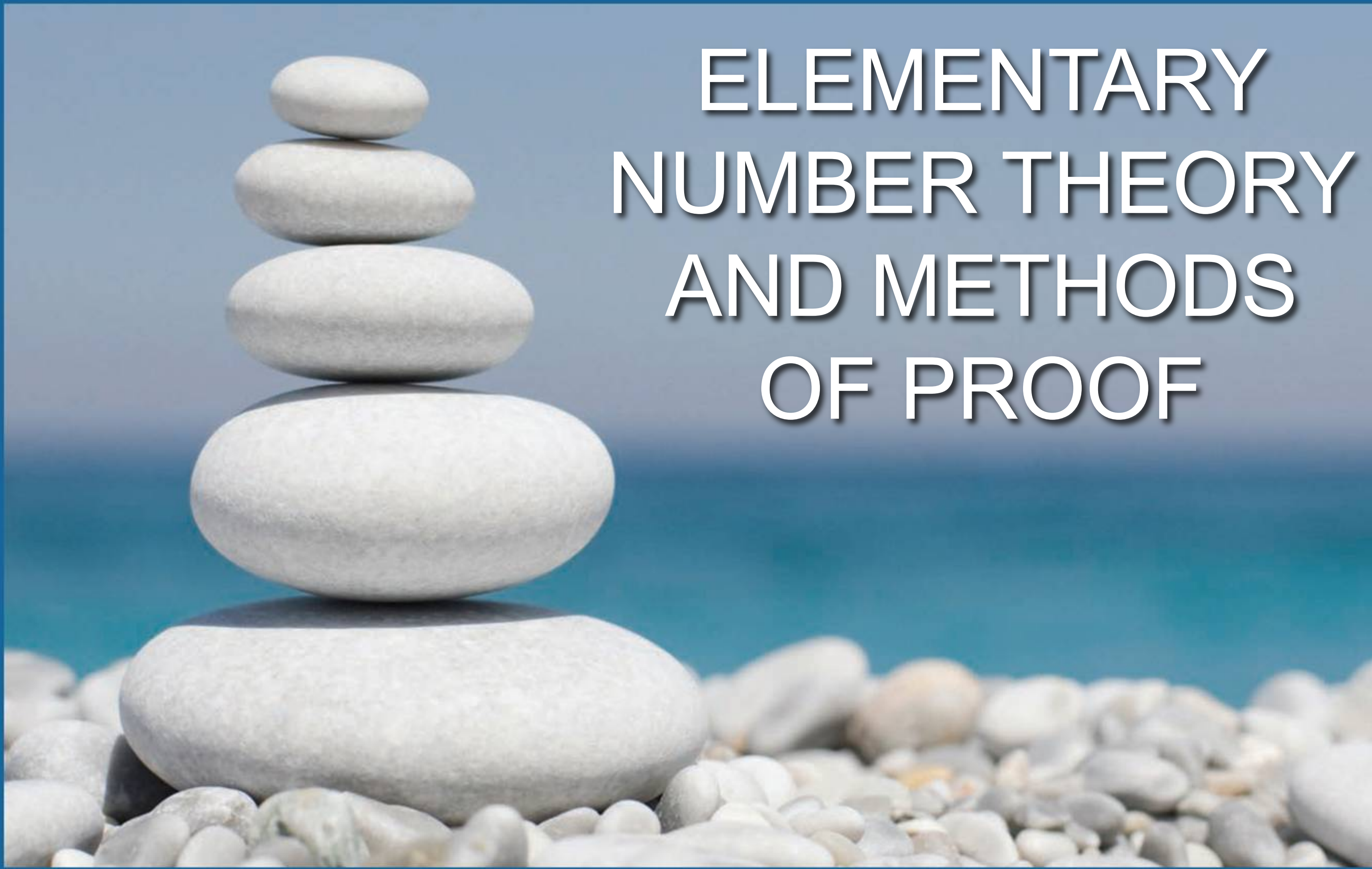
CHAPTER 4



ELEMENTARY NUMBER THEORY AND METHODS OF PROOF

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SECTION 4.6

Indirect Argument: Contradiction and Contraposition

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In a direct proof you start with the hypothesis of a statement and make one deduction after another until you reach the conclusion.

One kind of indirect proof, *by contradiction*, is based on the fact that either a statement is true or it is false but not both.

So if assuming that a given statement is false leads logically to a contradiction (or impossibility, or absurdity), then the given statement must be true.

This method of proof is known as *reductio ad impossible or reductio ad absurdum* because it relies on reducing a given assumption to an absurdity.

Indirect Argument: Contradiction and Contraposition

The point of departure for a proof by contradiction is the assumption that the statement to be proved is false. The goal is to reach a contradiction.

Method of Proof by Contradiction

1. Suppose the statement to be proved is false. That is, suppose that the negation of the statement is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Example 1 – *There Is No Greatest Integer*

Use proof by contradiction to show that there is no greatest integer.

Solution:

To prove that there is no object with this property, begin by supposing the negation: that there is an object with the property.

Starting Point: Suppose there is a greatest integer; call it N. This means that $N \geq n$, for all integers n.

To Show: This supposition leads logically to a contradiction.

Example 1 – *Solution*

cont'd

Theorem 4.6.1

There is no greatest integer.

Proof: [*We take the negation of the theorem and suppose it to be true.*] Suppose not. That is, suppose there is a greatest integer N. [*We must deduce a contradiction.*]

Then $N \geq n$, for every integer n. Let $M = N + 1$. Now M is an integer since it is a sum of integers. Also $M > N$ since $M = N + 1$. Thus M is an integer that is greater than N.

So N is the greatest integer and N is not the greatest integer, which is a contradiction. *[This contradiction shows that the supposition is false and, hence, that the theorem is true.]*

Argument by Contraposition

Argument by Contraposition

A second form of indirect argument, *argument by contraposition*, is based on the logical equivalence between a statement and its contrapositive.

To prove a statement by contraposition, you take the contrapositive of the statement, prove the contrapositive by a direct proof, and conclude that the original statement is true.

Argument by Contraposition

Method of Proof by Contraposition

1. Express the statement to be proved in the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x).$$

   (This step may be done mentally.)

2. Rewrite this statement in the contrapositive form

$$\forall x \text{ in } D, \text{ if } Q(x) \text{ is false then } P(x) \text{ is false.}$$

   (This step may also be done mentally.)

3. Prove the contrapositive by a direct proof.

   a. Suppose x is a (particular but arbitrarily chosen) element of D such that $Q(x)$ is false.

   b. Show that $P(x)$ is false.

Example 4 – *If the Square of an Integer Is Even, Then the Integer Is Even*

Prove that for all integers n, if n^2 is even then n is even.

Solution:

First form the contrapositive of the statement to be proved.

Contrapositive: For all integers n, if n is not even then n^2 is not even.

Fact: Any integer is even or odd, so any integer that is not even is odd. (This can be proved either by contradiction or using the quotient-remainder theorem with $d = 2$).

Example 4 – *Solution*

cont' d

Proof (by contraposition):

Suppose n is any odd integer. *[We must show that n^2 is odd.]* By definition of odd, $n = 2k + 1$ for some integer k. By substitution and algebra,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

But $2k^2 + 2k$ is an integer because products and sums of integers are integers.

So $n^2 = 2 \cdot (\text{an integer}) + 1$, and thus, by definition of odd, n^2 is odd *[as was to be shown].*

Relation between Proof by Contradiction and Proof by Contraposition

Relation between Proof by Contradiction and Proof by Contraposition

Observe that any proof by contraposition can be recast in the language of proof by contradiction.

The statement to be proved is of the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

Relation between Proof by Contradiction and Proof by Contraposition

Proof by contraposition: you suppose you are given an arbitrary element x of D such that $\sim Q(x)$. You then show that $\sim P(x)$. This is illustrated in Figure 4.6.1.

$$\boxed{\begin{array}{l}\text{Suppose } x \text{ is an arbitrary}\\ \text{element of } D \text{ such that } \sim Q(x).\end{array}} \xrightarrow{\text{sequence of steps}} \boxed{\sim P(x)}$$

Proof by Contraposition

To rewrite the proof as a proof by contradiction, you suppose there is an x in D such that $P(x)$ and $\sim Q(x)$.

$$\boxed{\begin{array}{l}\text{Suppose } \exists x \text{ in } D \text{ such}\\ \text{that } P(x) \text{ and } \sim Q(x).\end{array}} \xrightarrow{\text{same sequence of steps}} \boxed{\begin{array}{l}\text{Contradiction:}\\ P(x) \text{ and } \sim P(x)\end{array}}$$

Proof by Contradiction

Relation between Proof by Contradiction and Proof by Contraposition

As an example, here is a proof by contradiction of Proposition 4.6.4, namely that for any integer *n*, if n^2 is even then *n* is even.

Proposition 4.6.4

For all integers n, if n^2 is even then n is even.

Proof (by contradiction):
[We take the negation of the theorem and suppose it to be true.] Suppose not. That is, suppose there is an integer *n* such that n^2 is even and *n* is odd. *[We must deduce a contradiction.]*

Relation between Proof by Contradiction and Proof by Contraposition

Since n is odd, by definition of odd, $n = 2k + 1$ for some integer k. By substitution and algebra:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

But $2k^2 + 2k$ is an integer because products and sums of integers are integers.

So $n^2 = 2 \cdot (\text{an integer}) + 1$, and thus, by definition of odd, n^2 is odd. Therefore, n^2 is both even and odd.

Relation between Proof by Contradiction and Proof by Contraposition

This contradicts Theorem 4.6.2, which states that no integer can be both even and odd.

[This contradiction shows that the supposition is false and, hence, that the proposition is true.]

Relation between Proof by Contradiction and Proof by Contraposition

Note that when you use proof by contraposition, you know exactly what conclusion you need to show, namely the negation of the hypothesis; whereas in proof by contradiction, it may be difficult to know what contradiction to head for.

On the other hand, when you use proof by contradiction, once you have deduced any contradiction whatsoever, you are done.

Relation between Proof by Contradiction and Proof by Contraposition

Contraposition can be used only for a specific class of statements—those that are universal and conditional.

Thus, any statement that can be proved by contraposition can be proved by contradiction. But the converse is not true.

Statements such as "$\sqrt{2}$ is irrational" can be proved by contradiction but not by contraposition.