CHAPTER 7

FUNCTIONS

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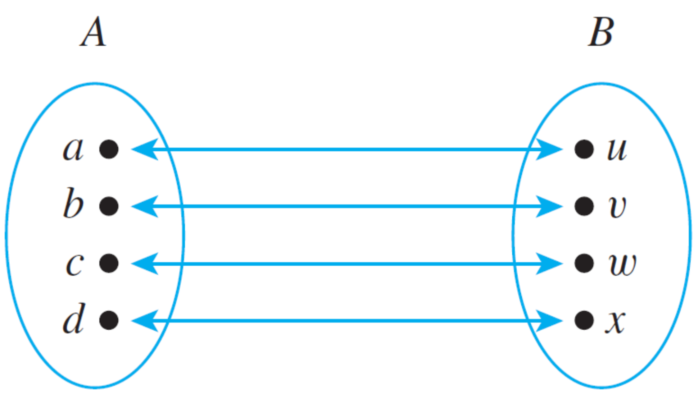
SECTION 7.4

Cardinality with Applications to Computability

Cardinality of Sets

We say that two finite sets whose elements can be paired by a one-to-one correspondence have the *same size*. This is illustrated by the following diagram.



The elements of set A can be put into one-to-one correspondence with the elements of B.

• Definition

Let A and B be any sets. **A has the same cardinality as B** if, and only if, there is a one-to-one correspondence from A to B. In other words, A has the same cardinality as B if, and only if, there is a function f from A to B that is one-to-one and onto.

Cardinality of Sets

Now a **finite set** is one that has no elements at all or that can be put into one-to-one correspondence with a set of the form $\{1, 2, \ldots, n\}$ for some positive integer n.

By contrast, an **infinite set** is a nonempty set that cannot be put into one-to-one correspondence with $\{1, 2, \ldots, n\}$ for any positive integer n.

Properties of Cardinality

The following theorem gives some basic properties of cardinality, most of which follow from statements proved earlier about one-to-one and onto functions.

Theorem 7.4.1 Properties of Cardinality

For all sets A, B, and C:

a. **Reflexive property of cardinality:** A has the same cardinality as A.

b. **Symmetric property of cardinality:** If A has the same cardinality as B, then B has the same cardinality as A.

c. **Transitive property of cardinality:** If A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C.

Cardinality of Infinite Sets

The following example illustrates a very important property of infinite sets—namely, that an infinite set can have the same cardinality as a proper subset of itself.

This property is sometimes taken as the definition of infinite sets.

The example shows that even though it may seem reasonable to say that there are twice as many integers as there are even integers, the elements of the two sets can be matched up exactly, and so, according to the definition, the two sets have the same cardinality.

Example 1 – *An Infinite Set and a Proper Subset Can Have the Same Cardinality*

Let $2\mathbf{Z}$ be the set of all even integers. Prove that $2\mathbf{Z}$ and $\mathbf{Z}$ have the same cardinality.

Solution:

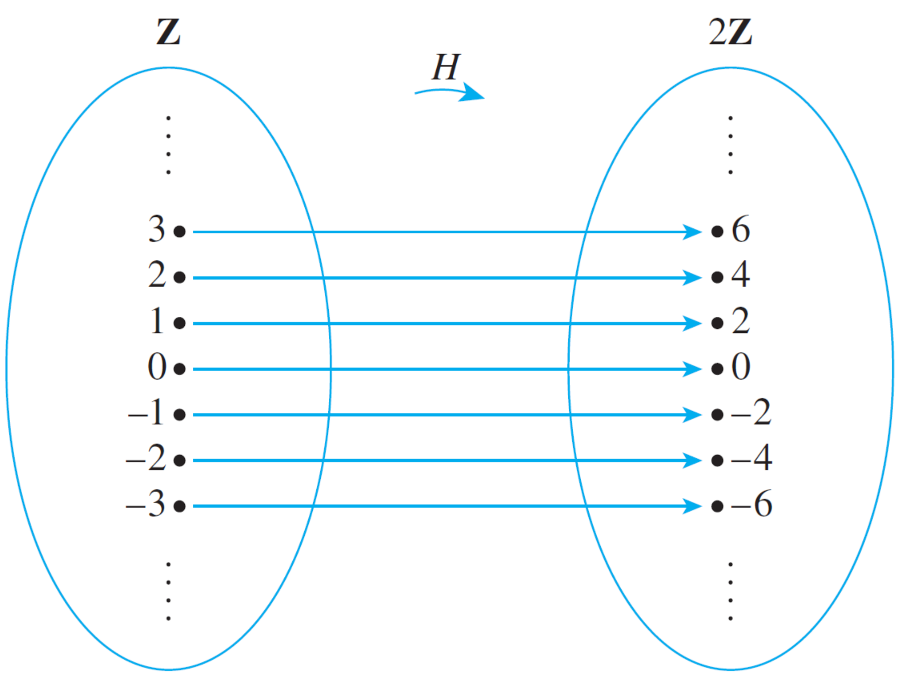
Consider the function H from $\mathbf{Z}$ to $2\mathbf{Z}$ defined as follows:

$$H(n) = 2n \qquad \text{for all } n \in \mathbf{Z}.$$

Example 1 – *Solution*

cont'd

A (partial) arrow diagram for H is shown below.



Example 1 – *Solution*

cont'd

To show that H is one-to-one, suppose $H(n_1) = H(n_2)$ for some integers n_1 and n_2.

Then, by definition of H, $\mathbf{2}\boldsymbol{n}_{\mathbf{1}} = \mathbf{2}\boldsymbol{n}_{\mathbf{2}}$ and dividing both sides by 2 gives $\boldsymbol{n}_{\mathbf{1}} = \boldsymbol{n}_{\mathbf{2}}$. Hence H is one-to-one.

To show that H is onto, suppose m is any element of $2\mathbf{Z}$. Then m is an even integer, and so $\boldsymbol{m} = \mathbf{2}\boldsymbol{k}$ for some integer k.

It follows that $\boldsymbol{H}(\boldsymbol{k}) = \mathbf{2}\boldsymbol{k} = \boldsymbol{m}$. Thus there exists k in $\mathbf{Z}$ with $H(k) = m$, and hence H is onto.

Therefore, by definition of cardinality, $\mathbf{Z}$ and $2\mathbf{Z}$ have the same cardinality.

Countable Sets

Countable Sets

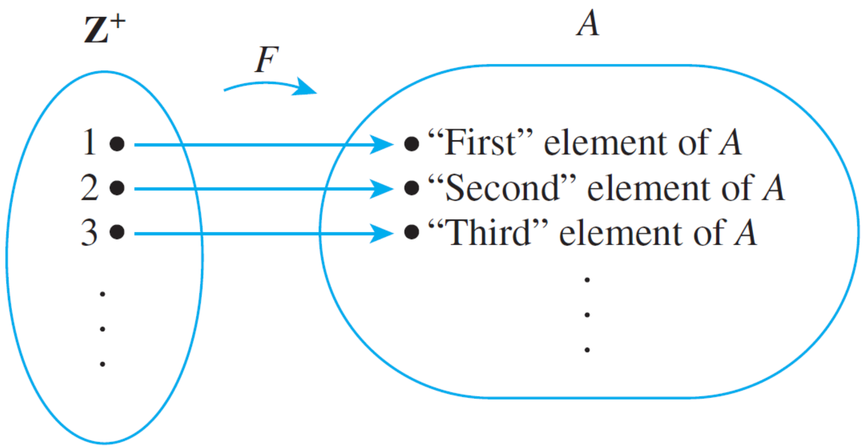
The set $\mathbf{Z}^+$ of counting numbers {1, 2, 3, 4, . . .} is, in a sense, the most basic of all infinite sets.

A set *A* having the same cardinality as this set is called ***countably infinite*.**

The reason is that the one-to-one correspondence between the two sets can be used to "count" the elements of *A*: If *F* is a one-to-one and onto function from $\mathbf{Z}^+$ to *A*, then *F*(1) can be designated as the first element of *A*, *F*(2) as the second element of *A*, *F*(3) as the third element of *A*, and so forth.

Countable Sets

This is illustrated graphically in Figure 7.4.1.



"Counting" a Countably Infinite Set

Figure 7.4.1

Because F is one-to-one, no element is ever counted twice, and because it is onto, every element of A is counted eventually.

Countable Sets

• Definition

A set is called **countably infinite** if, and only if, it has the same cardinality as the set of positive integers $\mathbf{Z}^+$. A set is called **countable** if, and only if, it is finite or countably infinite. A set that is not countable is called **uncountable.**

Example 2 – *Countability of Z, the Set of All Integers*

Show that the set **Z** of all integers is countable.

Solution:

The set **Z** of all integers is certainly not finite, so if it is countable, it must be because it is countably infinite.

To show that **Z** is countably infinite, find a function from the positive integers $\mathbf{Z}^+$ to **Z** that is one-to-one and onto.

Example 2 – *Solution*

cont’d

Looked at in one light, this contradicts common sense; judging from the diagram below, there appear to be more than twice as many integers as there are positive integers.

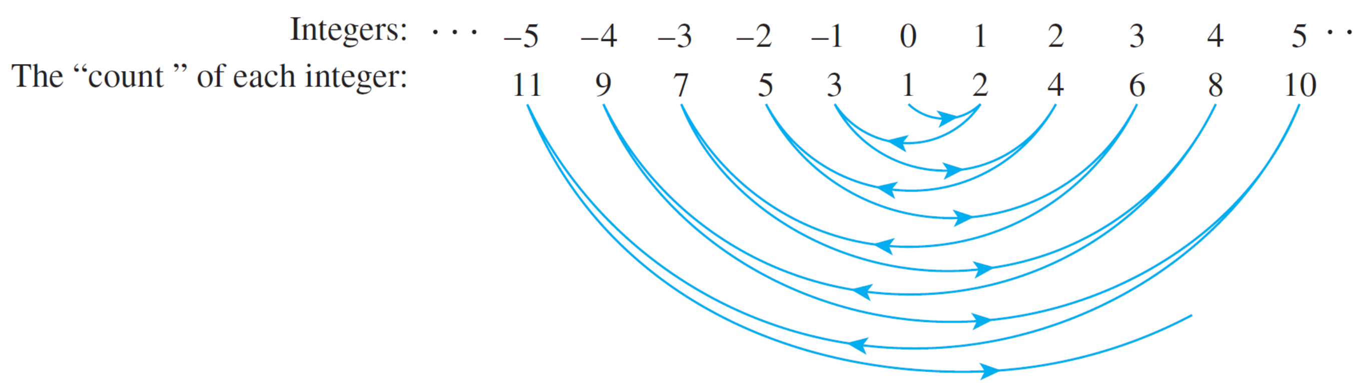
$$\underbrace{\cdots -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad \overbrace{1 \quad 2 \quad 3 \quad 4 \quad 5 \cdots}^{\text{positive integers}}}_{\text{all integers}}$$

But you were alerted that results in this section might be surprising. Try to think of a way to “count” the set of all integers anyway.

Example 2 – *Solution*

cont'd

The trick is to start in the middle and work outward systematically. Let the first integer be 0, the second 1, the third –1, the fourth 2, the fifth –2, and so forth as shown in Figure 7.4.2, starting at 0 and swinging outward in back-and-forth arcs from positive to negative integers and back again, picking up one additional integer at each swing.



"Counting" the Set of All Integers

Figure 7.4.2

Example 2 – *Solution*

cont'd

It is clear from the diagram that no integer is counted twice (so the function is one-to-one) and every integer is counted eventually (so the function is onto).

Consequently, this diagram defines a function from $\mathbf{Z}^+$ to $\mathbf{Z}$ that is one-to-one and onto.

Even though in one sense there seem to be more integers than positive integers, the elements of the two sets can be paired up one for one.

It follows by definition of cardinality that $\mathbf{Z}^+$ has the same cardinality as $\mathbf{Z}$. Thus $\mathbf{Z}$ is countably infinite and hence countable.

Example 2 – *Solution*

cont'd

The diagrammatic description of the previous function is acceptable as given. You can check, however, that the function can also be described by the explicit formula

$$F(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is an even positive integer} \\[2ex] -\dfrac{n-1}{2} & \text{if } n \text{ is an odd positive integer.} \end{cases}$$

The Search for Larger Infinities:
The Cantor Diagonalization Process

Example 4 – *The Set of All Positive Rational Numbers Is Countable*

Show that the set $\mathbf{Q}^+$ of all positive rational numbers is countable.

Solution:

Display the elements of the set $\mathbf{Q}^+$ of positive rational numbers in a grid as shown in Figure 7.4.3

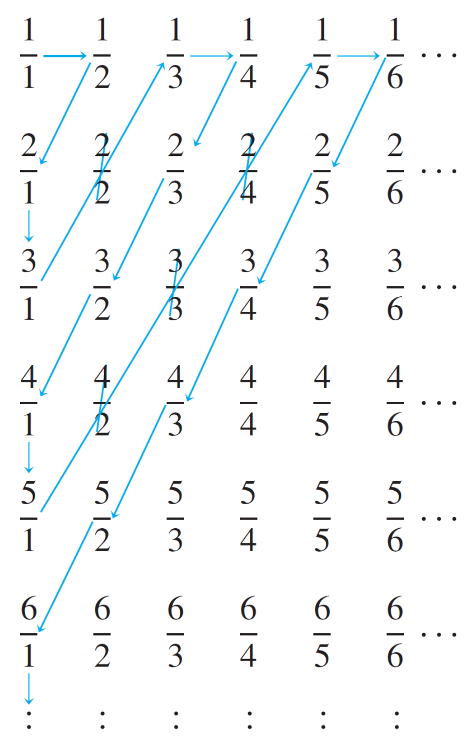


Figure 7.4.3

Example 4 – *Solution*

cont' d

Define a function F from $\mathbf{Z}^+$ to $\boldsymbol{Q}^+$ by starting to count at $\frac{1}{1}$ and following the arrows as indicated, skipping over any number that has already been counted.

To be specific: Set $F(1) = \frac{1}{1}$, $F(2) = \frac{1}{2}$, $F(3) = \frac{2}{1}$ and $F(4) = \frac{3}{1}$.

Then skip $\frac{2}{2}$ since $\frac{2}{2} = \frac{1}{1}$, which was counted first.

After that, set $F(5) = \frac{1}{3}$, $F(6) = \frac{1}{4}$, $F(7) = \frac{2}{3}$, $F(8) = \frac{3}{2}$, $F(9) = \frac{4}{1}$, and $F(10) = \frac{5}{1}$.

Example 4 – *Solution*

cont'd

Then skip $\frac{4}{2}$, $\frac{3}{3}$, and $\frac{2}{4}$ (since $\frac{4}{2} = \frac{2}{1}$, $\frac{3}{3} = \frac{1}{1}$, and $\frac{2}{4} = \frac{1}{2}$) and set $F(11) = \frac{1}{5}$.

Continue in this way, defining $F(n)$ for each positive integer n.

Example 4 – *Solution*

cont'd

The function F is onto.

Every positive rational number appears somewhere in the grid, and the counting procedure is set up so that every point in the grid is reached eventually.

The function F is one-to-one.

Skipping numbers that have already been counted ensures that no number is counted twice.

Consequently, F is a function from $\mathbf{Z}^+$ to $\mathbf{Q}^+$ that is one-to-one and onto, and so $\mathbf{Q}^+$ is countably infinite and hence countable.

The Search for Larger Infinities: The Cantor Diagonalization Process

In 1874 the German mathematician Georg Cantor achieved success in the search for a larger infinity by showing that the set of all real numbers is **uncountable**. His method of proof was somewhat complicated, however.

The uncountability of the set of all real numbers between 0 and 1 using a simpler technique introduced by Cantor in 1891 is called as the **Cantor diagonalization process**.

Over the intervening years, this technique and variations on it have been used to establish a number of important results in logic and the theory of computation.

The Search for Larger Infinities: The Cantor Diagonalization Process

Theorem 7.4.2 (Cantor)

The set of all real numbers between 0 and 1 is uncountable.

(Check the proof in the book)

Along with demonstrating the existence of an uncountable set, Cantor developed a whole arithmetic theory of infinite sets of various sizes. One of the most basic theorems of the theory states that any subset of a countable set is countable.

Theorem 7.4.3

Any subset of any countable set is countable.

The Search for Larger Infinities: The Cantor Diagonalization Process

An immediate consequence of Theorem 7.4.3 is the following corollary.

Corollary 7.4.4

Any set with an uncountable subset is uncountable.

Corollary 7.4.4 implies that the set of all real numbers is uncountable because the subset of numbers between 0 and 1 is uncountable.

In fact, the set of all real numbers has the same cardinality as the set of all real numbers between 0 and 1!

Example 5 – *The Cardinality of the Set of All Real Numbers*

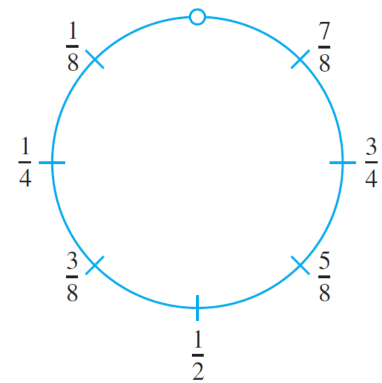
Show that the set of all real numbers has the same cardinality as the set of real numbers between 0 and 1.

Solution:

Let *S* be the open interval of real numbers between 0 and 1:

$$S = \{x \in \mathbf{R} \mid 0 < x < 1\}.$$

Imagine picking up *S* and bending it into a circle as shown in the right side. Since *S* does not include either endpoint 0 or 1, the top-most point of the circle is omitted from the drawing.

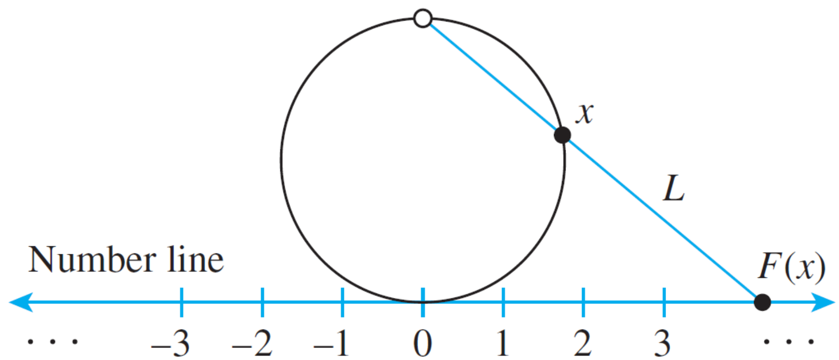


Example 5 – *Solution*

cont'd

Define a function $F: S \to \mathbf{R}$ as follows:

Draw a number line and place the interval, S, somewhat enlarged and bent into a circle, tangent to the line above the point 0. This is shown below.



Example 5 – *Solution*

cont'd

For each point x on the circle representing S, draw a straight line L through the topmost point of the circle and x.

Let $F(x)$ be the point of intersection of L and the number line. ($F(x)$ is called the *projection* of x onto the number line.)

Example 5 – *Solution*

cont'd

The function *F* is one-to-one.

It is clear from the geometry of the situation that distinct points on the circle go to distinct points on the number line, so *F* is one-to-one.

The function *F* is onto.

Given any point y on the number line, a line can be drawn through y and the top-most point of the circle. This line must intersect the circle at some point x, and, by definition, $y = F(x)$. Thus *F* is onto.

Hence *F* is a one-to-one correspondence from *S* to **R**, and so *S* and **R** have the same cardinality.

Greater Infinities

The infinity of the set of Real number is "**greater**" than the infinity of the set of positive Integers.

Can we have bigger infinities?

1. There is a one-to-one function between a set and its power set (map each element a to the singleton set {a});
2. They have a different cardinality.

The cardinality of any set is "less" than the cardinality of its power set.

We can create a larger and larger infinities! For example, you could begin with **Z**, the set of all integers, and take **Z**, P(**Z**), P(P(**Z**)), P(P(P(**Z**))), and so forth.

Application: Cardinality and Computability

Application: Cardinality and Computability

Knowledge of the countability and uncountability of certain sets can be used to answer a question of computability. We begin by showing that a certain set is countable.

Example 6 – *Countability of the Set of Computer Programs in a Computer Language*

Show that the set of all computer programs in a given computer language is countable.

Solution:

This result is a consequence of the fact that any computer program in any language can be regarded as a finite string of symbols in the (finite) alphabet of the language.

Given any computer language, let *P* be the set of all computer programs in the language. Either *P* is finite or *P* is infinite. If *P* is finite, then *P* is countable and we are done.

Example 6 – *Solution*

cont'd

If *P* is infinite, set up a binary code to translate the symbols of the alphabet of the language into strings of 0's and 1's. (For instance, either the seven-bit American Standard Code for Information Interchange, known as ASCII, or the eight-bit Extended Binary-Coded Decimal Interchange Code, known as EBCDIC, might be used.)

For each program in *P*, use the code to translate all the symbols in the program into 0's and 1's. Order these strings by length, putting shorter before longer, and order all strings of a given length by regarding each string as a binary number and writing the numbers in ascending order.

Example 6 – *Solution*

cont'd

Define a function $F: \mathbf{Z}^+ \to P$ by specifying that

$$F(n) = \text{the } n\text{th program in the list} \quad \text{for each } n \in \mathbf{Z}^+.$$

By construction, F is one-to-one and onto, and so P is countably infinite and hence countable.

As a simple example, suppose the following are all the programs in P that translate into bit strings of length less than or equal to 5:

$$10111,\ 11,\ 0010,\ 1011,\ 01,\ 00100,\ 1010,\ 00010.$$

Example 6 – *Solution*

Ordering these by length gives

$$\textit{length 2:}\ \ 11, 01$$

$$\textit{length 4:}\ \ 0010, 1011, 1010$$

$$\textit{length 5:}\ \ 10111, 00100, 00010$$

And ordering those of each given length by the size of the binary number they represent gives

$$\begin{aligned} 01 \quad &= F(1) \\ 11 \quad &= F(2) \end{aligned}$$

Example 6 – *Solution*

cont'd

$$
\begin{aligned}
0010 &= F(3) \\
1010 &= F(4) \\
1011 &= F(5) \\
00010 &= F(6) \\
00100 &= F(7) \\
10111 &= F(8)
\end{aligned}
$$

Note that when viewed purely as numbers, ignoring leading zeros, 0010 = 00010.

This shows the necessity of first ordering the strings by length before arranging them in ascending numeric order.