Logic: First Order Logic (Part I)

Alessandro Artale

Free University of Bozen-Bolzano
Faculty of Computer Science
http://www.inf.unibz.it/˜artale

Descrete Mathematics and Logic — BSc course

Thanks to Prof. Enrico Franconi for provoding the slides

Motivation

- We can already do a lot with propositional logic.
- But it is unpleasant that we cannot access the *structure* of atomic sentences.
- Atomic formulas of propositional logic are *too atomic* – they are just statement which may be true or false but which have no internal structure.
- In *First Order Logic* (FOL) the atomic formulas are interpreted as statements about *relationships between objects*.

Predicates and Constants

Let's consider the statements:

- *Mary is female*
  John is male
  Mary and John are siblings

In propositional logic the above statements are atomic propositions:

- `Mary-is-female`
  `John-is-male`
  `Mary-and-John-are-siblings`

In FOL atomic statements use predicates, with constants as argument:

- `Female(mary)`
  `Male(john)`
  `Siblings(mary,john)`

Variables and Quantifiers

Let's consider the statements:

- *Everybody is male or female*
- *A male is not a female*

In FOL, predicates may have variables as arguments, whose value is bounded by quantifiers:

- $\forall \mathtt{x}.\ \mathtt{Male}(\mathtt{x}) \vee \mathtt{Female}(\mathtt{x})$
- $\forall \mathtt{x}.\ \mathtt{Male}(\mathtt{x}) \rightarrow \neg \mathtt{Female}(\mathtt{x})$

Deduction (why?):

- *Mary is not male*
- $\neg \mathtt{Male}(\mathtt{mary})$

Functions

Let's consider the statement:

- *The father of a person is male*

In FOL objects of the domain may be denoted by functions applied to (other) objects:

- $\forall x.\ \mathtt{Male}(\mathtt{father}(x))$

Syntax of FOL: Terms and Atomic Sentences

Countably infinite **supply of symbols** (*signature*):

- variable symbols: $x, y, z, \ldots$
  n-ary function symbols: $f, g, h, \ldots$
  individual constants: $a, b, c, \ldots$
  n-ary predicate symbols: $P, Q, R, \ldots$

Terms:	t	$\rightarrow$	x	variable
		$\mid$	a	constant
		$\mid$	$f(t_1, \ldots, t_n)$	function application

Ground terms: terms that do not contain variables

Formulas:	ϕ	$\rightarrow$	$P(t_1, \ldots, t_n)$	atomic formulas

E.g., $Brother(kingJohn, richardTheLionheart)$

$> (length(leftLegOf(richard)), length(leftLegOf(kingJohn)))$

Syntax of FOL Formulas

$$
\begin{array}{llll}
\textbf{Formulas:} & \phi, \psi \rightarrow & P(t_1, \ldots, t_n) & \text{atomic formulas} \\
& \mid & \bot & \text{false} \\
& \mid & \top & \text{true} \\
& \mid & \neg\phi & \text{negation} \\
& \mid & \phi \land \psi & \text{conjunction} \\
& \mid & \phi \lor \psi & \text{disjunction} \\
& \mid & \phi \rightarrow \psi & \text{implication} \\
& \mid & \phi \leftrightarrow \psi & \text{equivalence} \\
& \mid & \forall x.\ \phi & \textit{universal quantification} \\
& \mid & \exists x.\ \phi & \textit{existential quantification}
\end{array}
$$

E.g. Everyone in England is smart: $\forall x.\ In(x, england) \rightarrow Smart(x)$

Someone in France is smart: $\exists x.\ In(x, france) \land Smart(x)$

Grounding FOL Formulas

- A ground term is a term which does not contain any variable. E.g., $\texttt{succ}(1, 2)$ is a ground function.
- A ground atomic formula is an atomic formula, all of whose terms are ground.
  E.g., $\texttt{Sibling}(\mathit{kingJohn}, \mathit{richard})$ is a ground atom.
- A ground literal is a ground atomic formula or the negation of one.
- A ground formula is a quantifier-free formula all of whose atomic formulas are ground.
  E.g.,
  $\texttt{Sibling}(\mathit{kingJohn}, \mathit{richard}) \rightarrow \texttt{Sibling}(\mathit{richard}, \mathit{kingJohn})$.

Summary of Syntax of FOL

- Terms
  - variables
  - constants
  - functions
- Literals
  - atomic formulas
    - relation (predicate)
  - negation of atomic formulas
- Well formed formulas
  - truth-functional connectives
  - existential and universal quantifiers

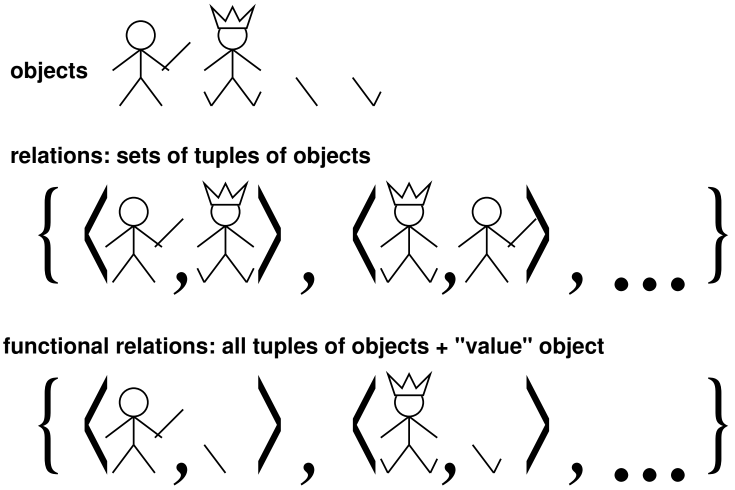
Semantics of FOL: Intuitions

- Just like in propositional logic, a (complex) FOL formula may be true (or false) with respect to a given interpretation.
- An interpretation specifies referents for
  constant symbols $\rightarrow$ **objects**
  predicate symbols $\rightarrow$ **relations**
  function symbols $\rightarrow$ **functional relations**
- An atomic sentence $P(t_1, \ldots, t_n)$ is true in a given interpretation

  iff

  the *objects* referred to by $t_1, \ldots, t_n$ are in the *relation* referred to by the predicate P.
- An interpretation in which a formula is true is called a *model* for the formula.

Models for FOL: Example



Semantic of FOL: Interpretations

Interpretation: $\mathcal{I} = \langle \mathbf{D}, \cdot^{\mathcal{I}} \rangle$ where $\mathbf{D}$ is an arbitrary non-empty set and $\mathcal{I}$ is a function that maps

- individual constants to elements of $\mathbf{D}$:
  $a^{\mathcal{I}} \in \mathbf{D}$
- n-ary function symbols to functions over $\mathbf{D}$:
  $f^{\mathcal{I}} \in [\mathbf{D}^n \rightarrow \mathbf{D}]$
- n-ary predicate symbols to relation over $\mathbf{D}$:
  $P^{\mathcal{I}} \subseteq \mathbf{D}^n$

Semantic of FOL: Ground Terms and Atoms

Interpretation of ground terms:

$$f(t_1, \ldots, t_n)^{\mathcal{I}} \quad = \quad f^{\mathcal{I}}({t_1}^{\mathcal{I}}, \ldots, {t_n}^{\mathcal{I}}) \ (\in \mathbf{D})$$

Satisfaction of ground atoms $P(t_1, \ldots, t_n)$:

$$\mathcal{I} \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle {t_1}^{\mathcal{I}}, \ldots, {t_n}^{\mathcal{I}} \rangle \in P^{\mathcal{I}}$$

Examples

$$\begin{array}{rcl}
\mathbf{D} & = & \{d_1, \ldots, d_n, n > 1\} \\
\mathtt{a}^{\mathcal{I}} & = & d_1 \\
\mathtt{b}^{\mathcal{I}} & = & d_2 \\
\mathtt{Block}^{\mathcal{I}} & = & \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} & = & \mathbf{D}
\end{array}$$

Examples

$$
\begin{array}{rcl}
\mathbf{D} & = & \{d_1, \ldots, d_n, n > 1\} \\
\mathtt{a}^{\mathcal{I}} & = & d_1 \\
\mathtt{b}^{\mathcal{I}} & = & d_2 \\
\mathtt{Block}^{\mathcal{I}} & = & \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} & = & \mathbf{D} \\
\mathcal{I} & \models & \mathtt{Red(b)} \\
\mathcal{I} & \not\models & \mathtt{Block(b)}
\end{array}
$$

Examples

$$
\begin{array}{rcl}
\mathbf{D} & = & \{d_1, \ldots, d_n, n > 1\} \\
\mathtt{a}^{\mathcal{I}} & = & d_1 \\
\mathtt{b}^{\mathcal{I}} & = & d_2 \\
\mathtt{Block}^{\mathcal{I}} & = & \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} & = & \mathbf{D} \\
\mathcal{I} & \models & \mathtt{Red(b)} \\
\mathcal{I} & \not\models & \mathtt{Block(b)}
\end{array}
$$

$$
\begin{array}{rcl}
\mathbf{D} & = & \{1, 2, 3, \ldots\} \\
1^{\mathcal{I}} & = & 1 \\
2^{\mathcal{I}} & = & 2 \\
 & \vdots & \\
\mathtt{Even}^{\mathcal{I}} & = & \{2, 4, 6, \ldots\} \\
\mathtt{succ}^{\mathcal{I}} & = & \{(1 \mapsto 2), (2 \mapsto 3), \ldots\}
\end{array}
$$

Examples

$$
\begin{aligned}
\mathbf{D} &= \{d_1, \ldots, d_n, n > 1\} \\
\mathtt{a}^{\mathcal{I}} &= d_1 \\
\mathtt{b}^{\mathcal{I}} &= d_2 \\
\mathtt{Block}^{\mathcal{I}} &= \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} &= \mathbf{D} \\
\mathcal{I} &\models \mathtt{Red(b)} \\
\mathcal{I} &\not\models \mathtt{Block(b)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{D} &= \{1, 2, 3, \ldots\} \\
1^{\mathcal{I}} &= 1 \\
2^{\mathcal{I}} &= 2 \\
&\vdots \\
\mathtt{Even}^{\mathcal{I}} &= \{2, 4, 6, \ldots\} \\
\mathtt{succ}^{\mathcal{I}} &= \{(1 \mapsto 2), (2 \mapsto 3), \ldots\} \\
\mathcal{I} &\not\models \mathtt{Even}(3) \\
\mathcal{I} &\models \mathtt{Even}(\mathtt{succ}(3))
\end{aligned}
$$

Let V be the set of all variables. A Variable Assignment is a function $\alpha\colon V \to \mathbf{D}$.

Notation: $\alpha[x/d]$ is a variable assignment identical to α except for the variable x mapped to d.

Interpretation of terms *under* $\mathcal{I}, \alpha$:

$$\begin{array}{rcl} x^{\mathcal{I},\alpha} & = & \alpha(x) \\ a^{\mathcal{I},\alpha} & = & a^{\mathcal{I}} \\ f(t_1, \ldots, t_n)^{\mathcal{I},\alpha} & = & f^{\mathcal{I}}({t_1}^{\mathcal{I},\alpha}, \ldots, {t_n}^{\mathcal{I},\alpha}) \end{array}$$

Satisfiability of atomic formulas:

$$\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) \quad \text{iff} \quad \langle {t_1}^{\mathcal{I},\alpha}, \ldots, {t_n}^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}}$$

Variable Assignment example

$$
\begin{array}{rcl}
\mathbf{D} & = & \{d_1, \ldots, d_n, n > 1\} \\
\mathtt{a}^{\mathcal{I}} & = & d_1 \\
\mathtt{b}^{\mathcal{I}} & = & d_2 \\
\mathtt{Block}^{\mathcal{I}} & = & \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} & = & \mathbf{D} \\
\alpha & = & \{(\mathtt{x} \mapsto d_1), (\mathtt{y} \mapsto d_2)\}
\end{array}
$$

$$
\begin{array}{rcl}
\mathcal{I}, \alpha & \models & \mathtt{Red(x)} \\
\mathcal{I}, \alpha[\mathtt{y}/d_1] & \models & \mathtt{Block(y)}
\end{array}
$$

Semantics of FOL: Satisfiability of formulas

A formula ϕ is satisfied by (*is true in*) an interpretation $\mathcal{I}$ under a variable assignment α, in symbols $\mathcal{I}, \alpha \models \phi$

$$
\begin{array}{rcl}
\mathcal{I}, \alpha \models P(t_1, \ldots, t_n) & \text{iff} & \langle {t_1}^{\mathcal{I},\alpha}, \ldots, {t_n}^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}} \\
\mathcal{I}, \alpha \models \neg\phi & \text{iff} & \mathcal{I}, \alpha \not\models \phi \\
\mathcal{I}, \alpha \models \phi \wedge \psi & \text{iff} & \mathcal{I}, \alpha \models \phi \text{ and } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \phi \vee \psi & \text{iff} & \mathcal{I}, \alpha \models \phi \text{ or } \mathcal{I}, \alpha \models \psi \\
\mathcal{I}, \alpha \models \forall x.\ \phi & \text{iff} & \text{for all } d \in \mathbf{D}: \\
 & & \mathcal{I}, \alpha[x/d] \models \phi \\
\mathcal{I}, \alpha \models \exists x.\ \phi & \text{iff} & \text{there exists a } d \in \mathbf{D}: \\
 & & \mathcal{I}, \alpha[x/d] \models \phi
\end{array}
$$

Examples

$$
\begin{aligned}
\mathbf{D} &= \{d_1, \ldots, d_n,\} \;\; n > 1 \\
\mathtt{a}^{\mathcal{I}} &= d_1 \\
\mathtt{b}^{\mathcal{I}} &= d_1 \\
\mathtt{c}^{\mathcal{I}} &= d_2 \\
\mathtt{Block}^{\mathcal{I}} &= \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} &= \mathbf{D} \\
\alpha &= \{(\mathrm{x} \mapsto d_1), (\mathrm{y} \mapsto d_2)\}
\end{aligned}
$$

1. $\mathcal{I}, \alpha \models \mathtt{Block(c)} \vee \neg\mathtt{Block(c)}$?

Examples

$$\begin{aligned}
\mathbf{D} &= \{d_1, \ldots, d_n,\} \;\; n > 1 \\
\mathtt{a}^{\mathcal{I}} &= d_1 \\
\mathtt{b}^{\mathcal{I}} &= d_1 \\
\mathtt{c}^{\mathcal{I}} &= d_2 \\
\mathtt{Block}^{\mathcal{I}} &= \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} &= \mathbf{D} \\
\alpha &= \{(\mathtt{x} \mapsto d_1), (\mathtt{y} \mapsto d_2)\}
\end{aligned}$$

1. $\mathcal{I}, \alpha \models \mathtt{Block(c)} \vee \neg\mathtt{Block(c)}$?
2. $\mathcal{I}, \alpha \models \mathtt{Block(x)} \rightarrow \mathtt{Block(x)} \vee \mathtt{Block(y)}$?

Examples

$$
\begin{array}{rcl}
\mathbf{D} & = & \{d_1, \ldots, d_n, \}\ \ n > 1 \\
\mathtt{a}^{\mathcal{I}} & = & d_1 \\
\mathtt{b}^{\mathcal{I}} & = & d_1 \\
\mathtt{c}^{\mathcal{I}} & = & d_2 \\
\mathtt{Block}^{\mathcal{I}} & = & \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} & = & \mathbf{D} \\
\alpha & = & \{(\mathtt{x} \mapsto d_1), (\mathtt{y} \mapsto d_2)\}
\end{array}
$$

1. $\mathcal{I}, \alpha \models \mathtt{Block(c)} \vee \neg\mathtt{Block(c)}$?
2. $\mathcal{I}, \alpha \models \mathtt{Block(x)} \rightarrow \mathtt{Block(x)} \vee \mathtt{Block(y)}$?
3. $\mathcal{I}, \alpha \models \forall \mathtt{x}.\ \mathtt{Block(x)} \rightarrow \mathtt{Red(x)}$?

Examples

$$
\begin{array}{rcl}
\mathbf{D} & = & \{d_1, \ldots, d_n,\} \;\; n > 1 \\
\mathtt{a}^{\mathcal{I}} & = & d_1 \\
\mathtt{b}^{\mathcal{I}} & = & d_1 \\
\mathtt{c}^{\mathcal{I}} & = & d_2 \\
\mathtt{Block}^{\mathcal{I}} & = & \{d_1\} \\
\mathtt{Red}^{\mathcal{I}} & = & \mathbf{D} \\
\alpha & = & \{(\mathtt{x} \mapsto d_1), (\mathtt{y} \mapsto d_2)\}
\end{array}
$$

1. $\mathcal{I}, \alpha \models \mathtt{Block(c)} \vee \neg \mathtt{Block(c)}$?
2. $\mathcal{I}, \alpha \models \mathtt{Block(x)} \rightarrow \mathtt{Block(x)} \vee \mathtt{Block(y)}$?
3. $\mathcal{I}, \alpha \models \forall \mathtt{x.}\ \mathtt{Block(x)} \rightarrow \mathtt{Red(x)}$?
4. $\Theta = \left\{ \begin{array}{l} \mathtt{Block(a)},\ \mathtt{Block(b)} \\ \forall \mathtt{x}\ (\mathtt{Block(x)} \rightarrow \mathtt{Red(x)}) \end{array} \right\}$
   $\mathcal{I}, \alpha \models \Theta$?

Example

Find a model of the formula:

$$\exists y.\ [\ P(y) \wedge \neg Q(y)\ ] \wedge \forall z. [\ P(z) \vee Q(z)\ ]$$

Example

Find a model of the formula:

$$\exists y.\ [\ P(y) \land \neg Q(y)\ ] \land \forall z.[\ P(z) \lor Q(z)\ ]$$

Possible Solution.

$\Delta = \{a, b\}$

$P^{\mathcal{I}} = \{a\}$

$Q^{\mathcal{I}} = \{b\}$

Satisfiability and Validity

An interpretation $\mathcal{I}$ is a **model** of ϕ *under* α, if

$$\mathcal{I}, \alpha \models \phi.$$

Similarly as in propositional logic, a formula ϕ can be **satisfiable**, **unsatisfiable**, **falsifiable** or **valid**—the definition is in terms of the pair $(\mathcal{I}, \alpha)$.
A formula ϕ is

- **satisfiable**, if there is some $(\mathcal{I}, \alpha)$ that satisfies ϕ;
- **unsatisfiable**, if ϕ is not satisfiable;
- **valid** (i.e., a **tautology**), if every $(\mathcal{I}, \alpha)$ is a model of ϕ;
- **falsifiable**, if there is some $(\mathcal{I}, \alpha)$ that does not satisfy ϕ.

Equivalence

Analogously, two formulas are **logically equivalent** ($\phi \equiv \psi$), if for all $(\mathcal{I}, \alpha)$ we have:

$$\mathcal{I}, \alpha \models \phi \quad \text{iff} \quad \mathcal{I}, \alpha \models \psi$$

Note: $\texttt{P(x)} \not\equiv \texttt{P(y)}$!

Free and Bound Variables

$$\forall x.\ (R(\boxed{y},\boxed{z}) \wedge \exists\ y.\ (\neg P(y,x) \vee R(y,\boxed{z})))$$

Variables in boxes are **free**; other variables are **bound**.

Definition. The free variables of a formula are inductively defined over the structure of formulas (structural induction):

$$\begin{array}{rcl}
\mathsf{free}(x) & = & \{x\} \\
\mathsf{free}(a) & = & \emptyset \\
\mathsf{free}(f(t_1,\ldots,t_n)) & = & \mathsf{free}(t_1) \cup \ldots \cup \mathsf{free}(t_n) \\
\mathsf{free}(P(t_1,\ldots,t_n)) & = & \mathsf{free}(t_1) \cup \ldots \cup \mathsf{free}(t_n) \\
\mathsf{free}(\neg\phi) & = & \mathsf{free}(\phi) \\
\mathsf{free}(\phi * \psi) & = & \mathsf{free}(\phi) \cup \mathsf{free}(\psi),\ * = \vee, \wedge, \ldots \\
\mathsf{free}(\forall x.\ \phi) & = & \mathsf{free}(\phi) - \{x\} \\
\mathsf{free}(\exists x.\ \phi) & = & \mathsf{free}(\phi) - \{x\}
\end{array}$$

Open and Closed Formulas

- A formula is **closed** or a **sentence** if no free variables occurs in it. Viceversa, the formula is said **open**.
- **Note:** For closed formulas, the properties *logical equivalence, satisfiability, entailment* etc. do not depend on variable assignments: If the property holds for one variable assignment then it holds for all of them. Thus,
  - For closed formulas, the symbol α on the left hand side of the "$\models$" sign is omitted:

$$\mathcal{I} \models \phi$$

Note: Unless specified, in the following we consider closed formulas.

Entailment

Entailment is defined similarly as in propositional logic.

Definition. The formula ϕ is **logically implied** by a formula ψ, if ϕ is true in all models of ψ (symbolically, $\psi \models \phi$):

$$\psi \models \phi \quad \text{iff} \quad \mathcal{I} \models \phi, \quad \text{for all models } \mathcal{I} \text{ of } \psi$$

More Exercises

- $\models \forall x.\ (P(x) \vee \neg P(x))$

More Exercises

- $\models \forall x.\ (P(x) \vee \neg P(x))$
- $\exists x.\ [\ P(x) \wedge (P(x) \rightarrow Q(x))\ ] \models \exists x.\ Q(x)$

More Exercises

- $\models \forall x.\ (P(x) \vee \neg P(x))$
- $\exists x.\ [\ P(x) \wedge (P(x) \rightarrow Q(x))\ ] \models \exists x.\ Q(x)$
- $\models \neg[\exists x. \forall y.\ (\ P(x) \rightarrow Q(y)\ )\ ]$

More Exercises

- $\models \forall x.\ (P(x) \lor \neg P(x))$
- $\exists x.\ [\ P(x) \land (P(x) \to Q(x))\ ] \models \exists x.\ Q(x)$
- $\models \neg[\exists x.\forall y.\ (\ P(x) \to Q(y)\ )\ ]$
- $\exists y.\ [\ P(y) \land \neg Q(y)\ ] \land \forall z.[\ P(z) \lor Q(z)\ ]$ satisfiable

Equality

Equality is a special predicate.

Definition. Given two terms, t_1, t_2, $t_1 = t_2$ is true under a given interpretation, $\mathcal{I}, \alpha \models t_1 = t_2$, if and only if t_1 and t_2 refer to the same object:

$$t_1^{\mathcal{I},\alpha} = t_2^{\mathcal{I},\alpha}$$

Consider the following examples:

$\forall x.\ (*(sqrt(x), sqrt(x)) = x)$, is satisfiable

$2 = 2$, is valid

Definition of (full) *Sibling* in terms of *Parent*:

$$\forall x, y. Sibling(x, y) \leftrightarrow$$
$$(\neg(x = y) \land \exists m, f. \neg(m = f) \land Parent(m, x) \land Parent(f, x) \land$$
$$Parent(m, y) \land Parent(f, y)\ )$$

Universal quantification

- Everyone in England is smart:
  $\forall x.\ LivesIn(x, england) \rightarrow Smart(x)$
- $(\forall x.\ \phi)$ is equivalent to the *conjunction* of all possible *instantiations* of x in ϕ:

$$
\begin{array}{ll}
 & LivesIn(kingJohn, england) \rightarrow Smart(kingJohn) \\
\wedge & LivesIn(richard, england) \rightarrow Smart(richard) \\
\wedge & LivesIn(england, england) \rightarrow Smart(england) \\
\wedge & \ldots
\end{array}
$$

- **Note.** Typically, $\rightarrow$ is the main connective with $\forall$.
  Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x.\ LivesIn(x, england) \wedge Smart(x)$$

means "Everyone lives in England and everyone is smart"

Existential quantification

- Someone in France is smart:
  $\exists x.\ \mathit{LivesIn}(x, \mathit{france}) \land \mathit{Smart}(x)$
- $(\exists x.\ \phi)$ is equivalent to the *disjunction* of all possible *instantiations* of x in ϕ:

$$\begin{array}{ll} & \mathit{LivesIn}(\mathit{kingJohn}, \mathit{france}) \land \mathit{Smart}(\mathit{kingJohn}) \\ \lor & \mathit{LivesIn}(\mathit{richard}, \mathit{france}) \land \mathit{Smart}(\mathit{richard}) \\ \lor & \mathit{LivesIn}(\mathit{france}, \mathit{france}) \land \mathit{Smart}(\mathit{france}) \\ \lor & \ldots \end{array}$$

- **Note.** Typically, $\land$ is the main connective with $\exists$.
  Common mistake: using $\rightarrow$ as the main connective with $\exists$:

$$\exists x.\ \mathit{LivesIn}(x, \mathit{france}) \rightarrow \mathit{Smart}(x)$$

is true if there is anyone who is not in France!

Logical Equivalences in FOL

Commutativity

- $(\forall x.\, \forall y.\, \phi) \equiv (\forall y. \forall x.\, \phi)$
- $(\exists x.\, \exists y.\, \phi) \equiv (\exists y.\, \exists x.\, \phi)$
- $(\exists x.\, \forall y.\, \phi) \not\equiv (\forall y.\, \exists x.\, \phi)$

$\forall$ and $\exists$ commute only in one direction

- $\models (\exists x.\, \forall y.\, \phi) \rightarrow (\forall y.\, \exists x.\, \phi)$

$\exists x.\, \forall y.\, Loves(x, y)$
"There is a person who loves everyone in the world", then
$\forall y.\, \exists x.\, Loves(x, y)$
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other.

- $\forall x.\, Likes(x, iceCream) \equiv \neg \exists x.\, \neg Likes(x, iceCream)$
- $\exists x.\, Likes(x, broccoli) \equiv \neg \forall x.\, \neg Likes(x, broccoli)$

Quantification distributes if the variable is not free.

$$
\begin{array}{rcll}
(\forall x.\, \phi) \land \psi & \equiv & \forall x.\, (\phi \land \psi) & \text{if } x \text{ not free in } \psi \\
(\forall x.\, \phi) \lor \psi & \equiv & \forall x.\, (\phi \lor \psi) & \text{if } x \text{ not free in } \psi \\
(\exists x.\, \phi) \land \psi & \equiv & \exists x.\, (\phi \land \psi) & \text{if } x \text{ not free in } \psi \\
(\exists x.\, \phi) \lor \psi & \equiv & \exists x.\, (\phi \lor \psi) & \text{if } x \text{ not free in } \psi
\end{array}
$$

$\forall$ **distributes over** $\land$ **-** $\exists$ **distributes over** $\lor$

$$
\begin{array}{rcl}
\forall x.\, (\phi \land \psi) & \equiv & \forall x.\, \phi \land \forall x.\, \psi \\
\exists x.\, (\phi \lor \psi) & \equiv & \exists x.\, \phi \lor \exists x.\, \psi
\end{array}
$$

Logical Equivalences in FOL (cont.)

Quantification over Implication.

$$
\begin{array}{rcl}
\forall x.\,(\phi \rightarrow \psi(x)) \equiv & \phi \rightarrow \forall x.\,\psi(x) & \text{if } x \text{ is not free in } \phi \\
\forall x.\,(\phi(x) \rightarrow \psi) \equiv & (\exists x.\,\phi(x)) \rightarrow \psi & \text{if } x \text{ is not free in } \psi \\
\exists x.\,(\phi(x) \rightarrow \psi(x)) \equiv & (\forall x.\,\phi(x) \rightarrow \exists x.\,\psi(x)) &
\end{array}
$$

Exercises

Show the following:

- $\neg\forall x.\,\phi \equiv \exists x.\,\neg\phi$ (De Morgan)
- $\neg\exists x.\,\phi \equiv \forall x.\,\neg\phi$ (De Morgan)
- $\not\models (\forall y.\,\exists x.\,\phi) \rightarrow (\exists x.\,\forall y.\,\phi)$
- $\models \forall x.\,\phi \vee \forall x.\,\psi \rightarrow \forall x(\phi \vee \psi)$
- $\not\models \forall x(\phi \vee \psi) \rightarrow \forall x.\,\phi \vee \forall x.\,\psi$
- $\models \exists x(\phi \wedge \psi) \rightarrow \exists x.\,\phi \wedge \exists x.\,\psi$
- $\not\models \exists x.\,\phi \wedge \exists x.\,\psi \rightarrow \exists x(\phi \wedge \psi)$
- $\models (\exists x.\,\phi(x) \rightarrow \forall x.\,\psi(x)) \rightarrow \forall x(\phi(x) \rightarrow \psi(x))$
- $\models \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow (\exists x.\,\phi(x) \rightarrow \exists x.\,\psi(x))$
- $\models \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow (\forall x.\,\phi(x) \rightarrow \exists x.\,\psi(x))$

The Prenex Normal Form

Quantifier prefix + (quantifier free) matrix

$$\forall x_1 \forall x_2 \exists x_3 \dots \forall x_n \phi$$

1. Elimination of $\rightarrow$ and $\leftrightarrow$
2. push $\neg$ inwards
3. pull quantifiers outwards

E.g. $\neg \forall \mathtt{x}.\ ((\forall \mathtt{x}.\ \mathtt{p}(\mathtt{x})) \rightarrow \mathtt{q}(\mathtt{x}))$
$\neg \forall \mathtt{x}.\ (\neg(\forall \mathtt{x}.\ \mathtt{p}(\mathtt{x})) \vee \mathtt{q}(\mathtt{x}))$
$\exists \mathtt{x}.\ ((\forall \mathtt{x}.\ \mathtt{p}(\mathtt{x})) \wedge \neg \mathtt{q}(\mathtt{x}))$
and now?

Definition: renaming of variables. Let $\phi[x/t]$ be the formula ϕ where all occurrences of x have been replaced by the term t.

Lemma. Let y be a variable that does not occur in ϕ.
Then we have $\forall x \phi \equiv (\forall x \phi)[x/y]$ and $\exists x \phi \equiv (\exists x \phi)[x/y]$.

Theorem. There is an algorithm that computes for every formula its equivalent prenex normal form:

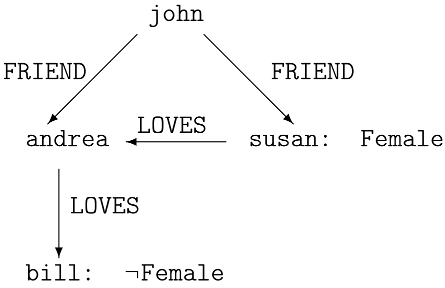
1. Rename bound variables;
2. Eliminate $\rightarrow$ and $\leftrightarrow$;
3. Push $\neg$ inwards;
4. Extract quantifiers outwards.

The Prenex Normal Form: Example

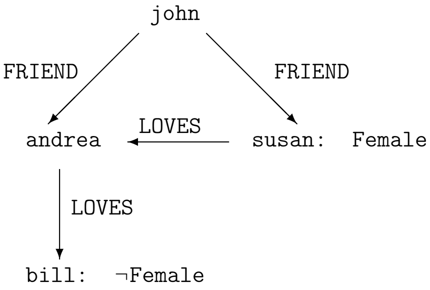
Original formula	$\exists x \forall y.\, p(x,y) \rightarrow \forall y \exists x.\, p(x,y)$
Rename bound variables	$\exists x \forall y.\, p(x,y) \rightarrow \forall w \exists z.\, p(z,w)$
Eliminate $\rightarrow$ and $\leftrightarrow$	$\neg \exists x \forall y.\, p(x,y) \vee \forall w \exists z.\, p(z,w)$
Push $\neg$ inwards	$\forall x \exists y.\, \neg p(x,y) \vee \forall w \exists z.\, p(z,w)$
Extract quantifiers outwards	$\forall x \exists y \forall w \exists z.\, \neg p(x,y) \vee p(z,w)$

FOL at work: reasoning by cases

$$\begin{aligned}\Theta = \ & \texttt{FRIEND(john,susan)} \wedge \\ & \texttt{FRIEND(john,andrea)} \wedge \\ & \texttt{LOVES(susan,andrea)} \wedge \\ & \texttt{LOVES(andrea,bill)} \wedge \\ & \texttt{Female(susan)} \wedge \\ & \neg\texttt{Female(bill)}\end{aligned}$$

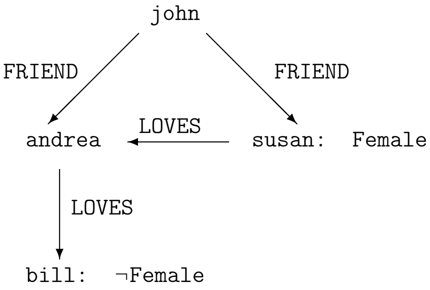


FOL at work: reasoning by cases (cont.)



Entailment: Does John have a female friend loving a male (i.e., not female) person?

FOL at work: reasoning by cases (cont.)



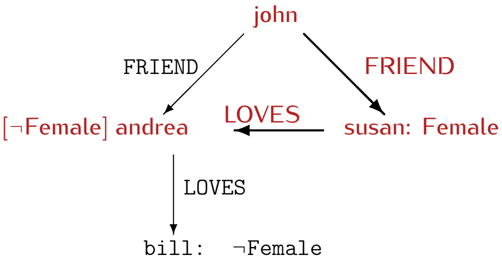
Entailment: Does John have a female friend loving a male (i.e., not female) person?

YES!

$\Theta \models \exists X, Y.$
$\texttt{FRIEND}(\texttt{john}, X) \land \texttt{Female}(X) \land \texttt{LOVES}(X, Y) \land \neg\texttt{Female}(Y)$

FOL at work: reasoning by cases (cont.)

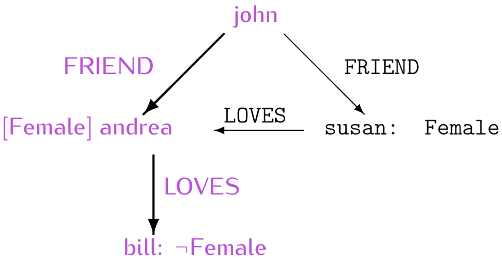
In all models where `andrea` is not a `Female`, then:



FRIEND(john,susan), Female(susan),
LOVES(susan,andrea), ¬Female(andrea)

FOL at work: reasoning by cases (cont.)

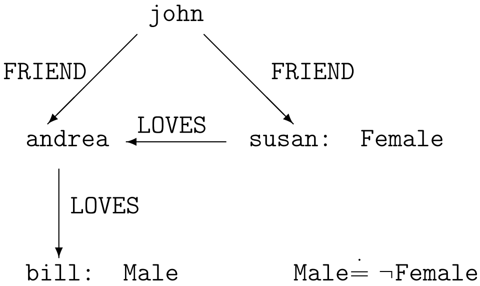
In all models where `andrea` is a `Female`, then:



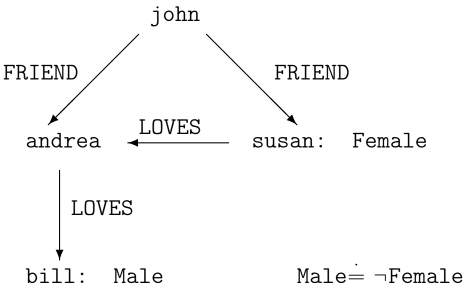
`FRIEND(john,andrea), Female(andrea),`
`LOVES(andrea,bill), ¬ Female(bill)`

Theories and Models

$$
\begin{aligned}
\Theta_1 = \ & \mathtt{FRIEND}(\mathtt{john}, \mathtt{susan}) \wedge \\
& \mathtt{FRIEND}(\mathtt{john}, \mathtt{andrea}) \wedge \\
& \mathtt{LOVES}(\mathtt{susan}, \mathtt{andrea}) \wedge \\
& \mathtt{LOVES}(\mathtt{andrea}, \mathtt{bill}) \wedge \\
& \mathtt{Female}(\mathtt{susan}) \wedge \\
& \mathtt{Male}(\mathtt{bill}) \wedge \\
& \forall X.\ \mathtt{Male}(X) \leftrightarrow \neg\mathtt{Female}(X)
\end{aligned}
$$



Theories and Models (cont.)



Entailment: Does John have a female friend loving a male person?

$$\Theta_1 \models \exists X, Y.\ \texttt{FRIEND}(\texttt{john}, X) \wedge \texttt{Female}(X) \wedge \texttt{LOVES}(X, Y) \wedge \texttt{Male}(Y)$$

Theories and Models (cont.)

$$\begin{aligned}\Theta = {} & \texttt{FRIEND(john,susan)} \wedge \\ & \texttt{FRIEND(john,andrea)} \wedge \\ & \texttt{LOVES(susan,andrea)} \wedge \\ & \texttt{LOVES(andrea,bill)} \wedge \\ & \texttt{Female(susan)} \wedge \\ & \neg\texttt{Female(bill)}\end{aligned}$$

$$\begin{aligned}\Theta_1 = {} & \texttt{FRIEND}(\texttt{john}, \texttt{susan}) \wedge \\ & \texttt{FRIEND}(\texttt{john}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{susan}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{andrea}, \texttt{bill}) \wedge \\ & \texttt{Female}(\texttt{susan}) \wedge \\ & \texttt{Male}(\texttt{bill}) \wedge \\ & \forall X.\ \texttt{Male}(X) \leftrightarrow \neg\texttt{Female}(X)\end{aligned}$$

Theories and Models (cont.)

$$\begin{aligned}\Theta = {} & \texttt{FRIEND(john,susan)} \wedge \\ & \texttt{FRIEND(john,andrea)} \wedge \\ & \texttt{LOVES(susan,andrea)} \wedge \\ & \texttt{LOVES(andrea,bill)} \wedge \\ & \texttt{Female(susan)} \wedge \\ & \neg\texttt{Female(bill)}\end{aligned}$$

$$\begin{aligned}\Theta_1 = {} & \texttt{FRIEND}(\texttt{john}, \texttt{susan}) \wedge \\ & \texttt{FRIEND}(\texttt{john}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{susan}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{andrea}, \texttt{bill}) \wedge \\ & \texttt{Female}(\texttt{susan}) \wedge \\ & \texttt{Male}(\texttt{bill}) \wedge \\ & \forall X.\ \texttt{Male}(X) \leftrightarrow \neg\texttt{Female}(X)\end{aligned}$$

$$\Delta = \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\}$$
$$\texttt{Female}^{\mathcal{I}} = \{\texttt{susan}\}$$

Theories and Models (cont.)

$$\begin{aligned}\Theta = & \ \texttt{FRIEND(john,susan)} \ \wedge \\ & \texttt{FRIEND(john,andrea)} \ \wedge \\ & \texttt{LOVES(susan,andrea)} \ \wedge \\ & \texttt{LOVES(andrea,bill)} \ \wedge \\ & \texttt{Female(susan)} \ \wedge \\ & \neg\texttt{Female(bill)}\end{aligned}$$

$$\begin{aligned}\Delta &= \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\} \\ \texttt{Female}^{\mathcal{I}} &= \{\texttt{susan}\}\end{aligned}$$

$$\begin{aligned}\Theta_1 = & \ \texttt{FRIEND}(\texttt{john}, \texttt{susan}) \wedge \\ & \texttt{FRIEND}(\texttt{john}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{susan}, \texttt{andrea}) \wedge \\ & \texttt{LOVES}(\texttt{andrea}, \texttt{bill}) \wedge \\ & \texttt{Female}(\texttt{susan}) \wedge \\ & \texttt{Male}(\texttt{bill}) \wedge \\ & \forall X.\ \texttt{Male}(X) \leftrightarrow \neg\texttt{Female}(X)\end{aligned}$$

$$\begin{aligned}\Delta^{\mathcal{I}_1} &= \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\} \\ \texttt{Female}^{\mathcal{I}_1} &= \{\texttt{susan}, \texttt{andrea}\} \\ \texttt{Male}^{\mathcal{I}_1} &= \{\texttt{bill}, \texttt{john}\}\end{aligned}$$

$$\begin{aligned}\Delta^{\mathcal{I}_2} &= \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\} \\ \texttt{Female}^{\mathcal{I}_2} &= \{\texttt{susan}\} \\ \texttt{Male}^{\mathcal{I}_2} &= \{\texttt{bill}, \texttt{andrea}, \texttt{john}\}\end{aligned}$$

$$\begin{aligned}\Delta^{\mathcal{I}_1} &= \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\} \\ \texttt{Female}^{\mathcal{I}_1} &= \{\texttt{susan}, \texttt{andrea}, \texttt{john}\} \\ \texttt{Male}^{\mathcal{I}_1} &= \{\texttt{bill}\}\end{aligned}$$

$$\begin{aligned}\Delta^{\mathcal{I}_2} &= \{\texttt{john}, \texttt{susan}, \texttt{andrea}, \texttt{bill}\} \\ \texttt{Female}^{\mathcal{I}_2} &= \{\texttt{susan}, \texttt{john}\} \\ \texttt{Male}^{\mathcal{I}_2} &= \{\texttt{bill}, \texttt{andrea}\}\end{aligned}$$

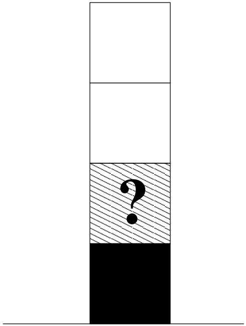
Theories and Models (cont.)

The following entailments hold:

$\Theta \not\models \texttt{Female(andrea)}$
$\Theta \not\models \neg\texttt{Female(andrea)}$

$\Theta_1 \not\models \texttt{Female(andrea)}$
$\Theta_1 \not\models \neg\texttt{Female(andrea)}$
$\Theta_1 \not\models \texttt{Male(andrea)}$
$\Theta_1 \not\models \neg\texttt{Male(andrea)}$

Exercise



Is it true that the top block is on a white block touching a black block?