Exercises for Logic 2015–2016

Propositional Logic: Weeks 8 and 9

Computer Science

Free University of Bozen-Bolzano
December 22, 2017

1 Equivalence

1.1 Equivalence

Use Definition 2.26 of equivalence "$\equiv$" of your textbook in order to prove the following equivalences:

1. $p \vee \top \equiv \top$;
2. $p \vee \bot \equiv p$;
3. $p \wedge \top \equiv p$;
4. $p \wedge \bot \equiv \bot$;
5. $p \vee \neg p \equiv \top$;
6. $p \wedge \neg p \equiv \bot$;
7. $\neg\neg p \equiv p$;
8. $p \to q \equiv \neg p \vee q$;
9. $p \to q \equiv \neg q \to \neg p$;
10. $\neg(p \vee q) \equiv \neg p \wedge \neg q$;
11. $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Then use the Substitution Theorem 2.34 of your textbook in order to generalise them to arbitrary propositional formulae.

1.2 Biimplication and Equivalence

Prove that the following formulae are valid building only the necessary rows of their truth table:

$$\Big(\big(p \wedge (q \vee r)\big) \leftrightarrow \big((p \wedge q) \vee (p \wedge r)\big)\Big)$$
$$\Big(\big(p \vee (q \wedge r)\big) \leftrightarrow \big((p \vee q) \wedge (p \vee r)\big)\Big)$$

Then use the above, Theorem 2.29 concerning biimplication and equivalence and the Substitution Theorem 2.34 of your textbook in order to justify the following equivalences:

$$\big(\varphi \wedge (\psi \vee \chi)\big) \equiv \big((\varphi \wedge \psi) \vee (\varphi \wedge \chi)\big)$$
$$\big(\varphi \vee (\psi \wedge \chi)\big) \equiv \big((\varphi \vee \psi) \wedge (\varphi \vee \chi)\big)$$

2 Entailment via Truth Tables or Semantic Arguments

2.1 Entailment: From Truth Tables to Formulae

Let $P = \{p, q, r\}$. Let φ, ψ, θ, χ be formulae of PL(P) with the following truth table:

p	q	r	φ	ψ	θ	χ
1	1	1	**1**	**0**	**0**	**1**
1	1	0	**0**	**0**	**0**	**0**
1	0	1	**1**	**1**	**1**	**0**
1	0	0	**1**	**0**	**0**	**1**
0	1	1	**0**	**0**	**1**	**0**
0	1	0	**1**	**1**	**1**	**1**
0	0	1	**1**	**0**	**1**	**0**
0	0	0	**1**	**1**	**0**	**0**

1. Is $\{(\varphi \leftrightarrow \neg\psi), \theta, \chi\}$ satisfiable? Explain your answer.
2. Determine whether $\{\varphi, \psi, \theta\} \models \chi$.
3. Determine whether $\models (\varphi \vee \psi \vee \theta \vee \chi)$.

2.2 Entailment via Semantic Arguments

Prove or refute (i.e., show a counterexample) each of the following assertions using interpretations (i.e., $\Theta \models \varphi$ iff *every* model of Θ is a model of φ) or truth tables:

1. $\Theta \cup \{\neg\varphi\}$ is unsatisfiable iff $\Theta \models \varphi$.
2. if $\Theta \models \varphi \wedge \varphi'$ then $\Theta \models \varphi$ (and $\Theta \models \varphi'$);
3. if $\Theta \models \varphi \vee \varphi'$ then $\Theta \models \varphi$;
4. if $\Theta \models \varphi$ then $\Theta \models \varphi \vee \varphi'$.

3 Validity and Satisfiability of a Formula via Tableaux

Note. We follow Algorithm 2.64 of the course textbook for building a tableau, with the following differences:

- We proceed as in Section 2.6.4 for building **analytic tableaux**;
- Each input formula is first transformed in an equivalent formula in Negation Normal Form (NNF), thus
- The only rules that will be used are the AND-rule and the OR-rule.

3.1 Tableaux for Validity

Consider the following problem: are the following formulae valid (true under all interpretations for its atoms, a.k.a, a tautology)? Decide on it with the (analytic) tableau procedure.

1. $(p \to q) \to (\neg q \to \neg p)$;
2. $(q \to p) \to p$;
3. $(\bot \wedge \neg\neg q) \vee q$.

3.2 Tableaux for Satisfiability

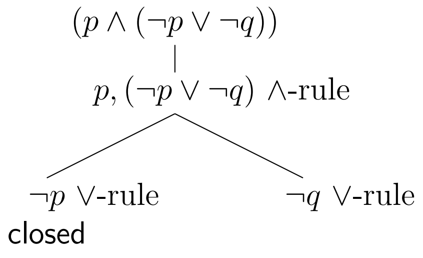
Use the tableau procedure and: (1) prove that the following formulae are satisfiable; (2) build a model for each of them.

1. $\neg(p \to (p \wedge q))$;
2. $(p \vee (p \wedge q))$ *(homework)*;
3. $(\bot \wedge \neg\neg q) \vee q$ *(homework)*;
4. $(p \vee q) \wedge (\neg p \vee \neg q)$;

Resolution of 1:

We first generate the Negation Normal Form (NNF) of the given formula:

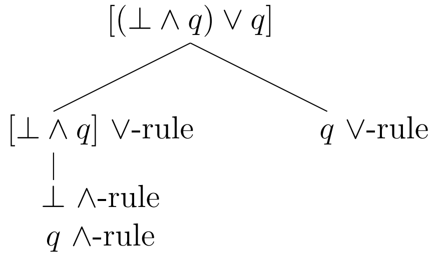
$\neg(p \to (p \wedge q)) \equiv \neg(\neg p \vee (p \wedge q)) \equiv (p \wedge (\neg p \vee \neg q))$


$(p \wedge (\neg p \vee \neg q))$
$p, (\neg p \vee \neg q)$ $\wedge$-rule
$\neg p$ $\vee$-rule
closed
$\neg q$ $\vee$-rule

The tableau has an open branch and thus the formula is satisfiable, A model is the interpretation I with $I(p) = 1$ and $I(q) = 0$, obtained from the open (right) branch.

Resolution of 3:

We first generate the Negation Normal Form (NNF) of the given formula:
$(\bot \wedge \neg\neg q) \vee q \equiv (\bot \wedge q) \vee q$



The tableau has an open branch and thus the formula is satisfiable, A model is the interpretation I with $I(q) = 1$, obtained from the open (right) branch.

3.3 Opitmisations for Tableaux

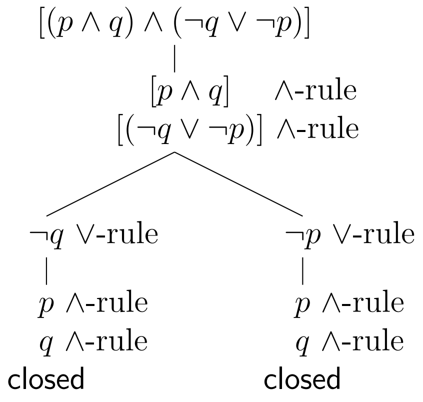
3.3.1 Selection Heuristics

Consider the following formula: $\neg((p \wedge q) \to (q \wedge p))$. Build two tableau for it: one in which you apply an OR-rule before an AND-rule; the other in which you do the vice-versa.

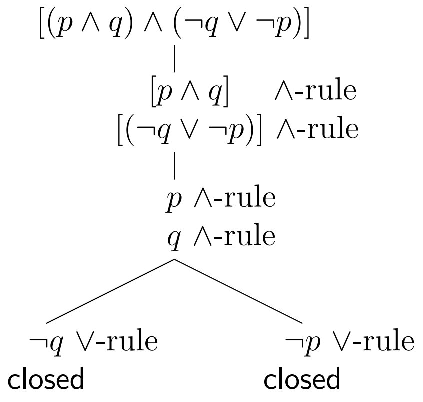
Resolution.

We first generate the Negation Normal Form (NNF) of the given formula:
$\neg((p \wedge q) \to (q \wedge p)) \equiv \neg(\neg(p \wedge q) \vee (q \wedge p)) \equiv (p \wedge q) \wedge (\neg q \vee \neg p)$

Applying first an OR-rule before an AND-rule, we get the following tableau.



Applying first an AND-rule before an OR-rule, we get the following tableau.


$[(p \wedge q) \wedge (\neg q \vee \neg p)]$
$[p \wedge q]$ $\wedge$-rule
$[(\neg q \vee \neg p)]$ $\wedge$-rule
p $\wedge$-rule
q $\wedge$-rule
$\neg q$ $\vee$-rule
$\neg p$ $\vee$-rule
closed
closed

The latter tableau is built with the following optimisation heuristics: choosing first nodes with AND (α) formulae, and then nodes with OR (β) formulae. In this manner, the same formulae are not repeated in different branches; rather, they occur as left labels above all those branching nodes.

3.3.2 Preprocessing Strategies

Consider the following problem. Which of the following formulae is a tautology (true under all interpretations for its atoms, valid)? Consider the equivalences in Ex. 1.1 from 1–7. Use these for pre-processing the following formulae into equivalent ones so as to reduce their length (as strings). Then apply the tableau procedure to decide on the above problem, if needed.

1. $(\top \vee \neg\neg q) \vee (q \wedge \neg q)$;

2. $(p \to (q \vee p)) \wedge (q \wedge \neg q)$.

4 Satisfiability of a Set of Formulae via Tableaux

4.1 Satisfiability of a Set of Formulae and Satisfiability of a Formula

Deciding on the satisfiability of $\{\varphi_1, \ldots, \varphi_n\}$ is equivalent to deciding on the satisfiability of $\bigwedge_{i=1}^{n} \varphi_i$, with $n \geq 2$. Prove it.

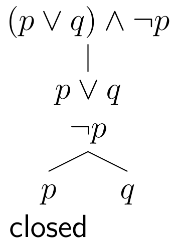
4.2 Satisfiability of a Set of Formulae via Tableaux

The equivalence in Exercise 4.1 and Corollary 2.68 of your textbook allow us to use tableaux for deciding on the satisfiability of a finite set of formulae.

Consider any $\{\varphi_1, \ldots, \varphi_n\} \subseteq \mathrm{PL}(P)$. Build a tableau for $\bigwedge_{i=1}^{n} \varphi_i$ with the tableau procedure in your textbook. If the tableau has an open branch, then $\{\varphi_1, \ldots, \varphi_n\}$ is satisfiable, and the open branch defines a model for the set; else the set is unsatisfiable.

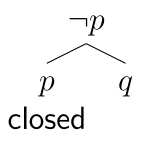
4.3 Verifying Set Satisfiability with Tableaux

Let us prove that the set $\{p \vee q, \neg p\}$ is satisfiable by building a tableau with an open branch for the formula $(p \vee q) \wedge \neg p$.



The rigth-more branch gives the interpretation $\mathcal{I}(q) = \mathtt{T}$ and $\mathcal{I}(p) = \mathtt{F}$ that satisfies the set.

Note 1. In general, to save space, we can silently apply the $\wedge$-rule when building a tableau for $\Theta = \{\varphi_1, \ldots, \varphi_n\}$, and write all the formulae of Θ one below the other, left labelling the root. With such expedient, the previous tableau would be shortened as follows.



5 Entailment via Tableaux

5.1 Duality of Satisfiability and Entailment via Negation

Having $\neg$ in the language, entailment and set (un)satisfiability are complementary problems in the following sense: $\Theta \models \varphi$ iff $\Theta \cup \{(\neg\varphi)\}$ is unsatisfiable. Prove it with the semantic argument—reasoning on the definition of interpretation.

Proof. Let us spell out what "$\Theta \models \varphi$" and "$\Theta \cup \{(\neg\varphi)\}$" mean.
(1) $\Theta \models \varphi$ means the following: if $\mathcal{I} \models \Theta$ then $\mathcal{I} \models \varphi$.
(2) $\Theta \cup \{(\neg\varphi)\}$ is unsatisfiable means the following: if $\mathcal{I} \models \Theta$ then $\mathcal{I} \not\models \neg\varphi$.
Therefore proving that

$$(1) \text{ iff } (2)$$

means proving that

$$\mathcal{I} \models \varphi \text{ iff } \mathcal{I} \not\models \neg\varphi$$

which is always true by definition of interpretation of a negated formula. □

5.2 Entailment and Validity

Deciding on $\{\varphi_1, \ldots, \varphi_n\} \models \varphi$ is equivalent to deciding on the unsatisfiability of $\bigwedge_{i=1}^{n} \varphi_i \wedge (\neg\varphi)$. Prove it.

Proof. It follows from Exercises 5.1 and 4.2. □

5.3 Entailment via Tableaux

The equivalence in Exercise 5.2 and Corollary 2.68 of your textbook allow us to use tableaux for deciding on entailment from a finite set of formulae.

Consider any $\{\varphi_1, \ldots, \varphi_n\} \cup \{\varphi\} \subseteq \mathrm{PL}(P)$. Build a tableau for $\bigwedge_{i=1}^{n} \varphi_i \wedge (\neg\varphi)$ with the tableau procedure in your textbook. If all the branches of the tableau are closed, then $\{\varphi_1, \ldots, \varphi_n\} \models \varphi$; else $\{\varphi_1, \ldots, \varphi_n\} \not\models \varphi$, and an open branch of the tableau defines a model of $\{\varphi_1, \ldots, \varphi_n\}$ that does not satisfy φ.

5.4 Verifying Entailment with Tableaux

Let us prove $p \wedge q \models p$ by building a tableau for $(p \wedge q) \wedge \neg p$ with only closed branches.

$$\begin{array}{c} (p \wedge q) \wedge \neg p \\ | \\ p \wedge q \\ \neg p \\ | \\ p \\ \text{closed} \end{array}$$

Note 2. With the expedient in Note 1, the previous tableau can be shortened as follows.

$$\begin{array}{c} p \wedge q \\ \neg p \\ | \\ p \\ \text{closed} \end{array}$$

6 Formalisation and Decisions

1. Consider the following argument.

> If Paul lives in Dublin, he lives in Ireland. Paul lives in Dublin. *Therefore* Paul lives in Ireland.

(i) Formalise the argument using a minimal propositional language; (ii) use the semantic argument (e.g., use interpretations), truth tables or tableaux in order to decide on the entailment; (iii) build a truth table or use interpretations or tableaux to decide on the satisfiability of the resulting set of formulae.

2. (*Homework*) Suppose you are given the following set of statements:

1. If the movie "Asterix" is not worth seeing then it was not made in Britain.
2. "Asterix" is worth seeing only if critic Chris reviews it.
3. The movie "Asterix" was not reviewed by Chris.
4. Therefore "Asterix" was not made in Britain.

Use the semantic argument (e.g., use interpretations), truth tables or tableaux in order to decide on the entailment.

3. Suppose that an island is inhabited by exactly two persons, Angelo and Roberto. Given this, consider the following argument.

> (1) If Angelo shaves an inhabitant then this shaves Roberto. (2) Moreover, if Roberto shaves an inhabitant then this does not shave Angelo. (3) Roberto does not shave Angelo. (3′) Roberto shaves Angelo.

(i) Formalise (1), (2), (3) and (3′) using a suitable propositional language.

(ii) build a truth table or use interpretations or tableaux to decide on the entailment problem $\{(1), (2)\} \models (3)$.

(iii) (*Homework*) build a truth table or use interpretations or tableaux to decide on the satisfiability of the set of formulae (1), (2), (3′).

(iv) Could you formalise (1), (2) and (3) in a propositional language if you did not know the number of inhabitants? Try using a first order language.