# Exercises for Discrete Maths

Discrete Maths

Teacher: Alessandro Artale
Teaching Assistants: Elena Botoeva, Daniele Porello
http://www.inf.unibz.it/~artale/DML/dml.htm

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**Computer Science**

Free University of Bozen-Bolzano

# (Structural) Induction

**1.** If $n$ people stand in a line, where $n \geq 2$, and the first (position 1) is a woman and the last (position $n$) is a man, then, somewhere in the line, there is a woman directly before a man. Call this property

*Front*$(k)$ = 'If in a line of $k$ people, the first one is a woman, and the $k$th one is a man, then there exists $j < k$ such that there is a woman in position $j$ and a man in position $j + 1$.'

**Proof.** **Basis Step:** $n = 2$: By assumption a woman is at position 1 and a man at position 2.

**Inductive Step:** Assume *Front*$(k)$ is true for some $k \geq 2$. We need to prove $Front(k + 1)$. Define:

$$Person(j) = \begin{cases} woman, & \text{if the person in position } j \text{ is a woman} \\ man, & \text{otherwise} \end{cases}$$

Now, we have that

$$Person(k) = woman \vee man$$

(we assume these concepts here for simplicity to be mutually exclusive and at least one of 'woman' or 'man' is true).

In case $Person(k) = man$, then, by inductive hypothesis, *Front*$(k)$ is true, i.e. there is a woman in front of a man somewhere between 1 and $k$, so *Front*$(k + 1)$ is true.

In case $Person(k) = woman$, since $Person(k + 1) = man$, then also *Front*$(k + 1)$ is true.

**2.** Let $S$ be the subset of the set of ordered pairs of integers defined recursively as:

Basis Step: $(0, 0) \in S$

Inductive Step: if $(a, b) \in S$ then $(a, b + 1), (a + 1, b + 1), (a + 2, b + 1) \in S$

a) List the first 4 elements:

$$\{(0, 0), (0, 1), (1, 1), (2, 1)\}$$

b) Use structural induction to show that

$$\forall (a, b) \in S. a \leq 2b$$

**Proof.** **Basis Step:** True for $(0, 0)$, i.e. $0 \leq 0$.

**Inductive Step:** Let $(a, b) \in S$. We show that for any of the three rules given above the property holds if it holds for $(a, b)$.

Rule 1: $(a,b) \longrightarrow (a,b+1)$. By induction hypothesis $a \leq 2b$. Therefore $a \leq 2b+2 = 2(b+1)$.

Rule 2: $a \leq 2b \implies a+1 \leq 2b+1 < 2b+2 = 2(b+1)$. So $a+1 \leq 2(b+1)$.

Rule 3: $a \leq 2b \implies a+2 \leq 2b+2 = 2(b+1)$.

This completes the inductive step.

Exercises for Section 6.1 (p. 351)

## Set Theory

**Exercise 31.** Let $A = \{1,2\}$ and $B = \{2,3\}$. Find the following power sets:

a) $\mathcal{P}(A \cap B) = \mathcal{P}(\{2\}) = \{\emptyset, \{2\}\}$

b) $\mathcal{P}(A) = \mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

c) $\mathcal{P}(A \cup B) = \mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

d) $\mathcal{P}(A \times B) = \mathcal{P}(\{1,2\} \times \{2,3\}) = \mathcal{P}(\{(1,2),(1,3),(2,2),(2,3)\}) =$

$$\Big\{\emptyset, \{(1,2)\}, \{(1,3)\}, \{(2,2)\}, \{(2,3)\},$$
$$\{(1,2),(1,3)\}, \{(1,2),(2,2)\}, \{(1,2),(2,3)\}, \{(1,3),(2,2)\}, \{(1,3),(2,3)\}, \{(2,2),(2,3)\},$$
$$\{(1,2),(1,3),(2,2)\}, \{(1,2),(1,3),(2,3)\}, \{(1,2),(2,2),(2,3)\}, \{(1,3),(2,2),(2,3)\},$$
$$\{(1,2),(1,3),(2,2),(2,3)\}\Big\}$$

**Note:** Alternative notation $\mathcal{P}(A) = 2^A$

**Exercise 33.** Find:

a) $\mathcal{P}(\emptyset)$ **Answer:** $\{\emptyset\}$

b) $\mathcal{P}(\mathcal{P}(\emptyset))$ **Answer:** $\{\emptyset, \{\emptyset\}\}$

c) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$ **Answer:** $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

**Exercise set 6.3 (p. 372). 17.** Show that $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$.

**Answer.** Suppose that $A \subseteq B$ and let $X \in \mathcal{P}(A)$. By definition, we have $X \subseteq A$, therefore (by transitivity of inclusion) $X \subseteq B$. So, by definition of powerset, $X \in \mathcal{P}(B)$.

**18.** Show that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ is false in general.

**Answer.** We construct a counter-example. Pick a set $A$ with a subset $\emptyset \neq X_A \subseteq A$ such that $X_A \nsubseteq B$ and a $z \in B$ with $z \notin A$ (this is allowed because we can freely choose $A$ and $B$ when constructing a counterexample). For instance, $A = \{1,2\}$ and $B = \{2,3\}$ and $X_A = \{1\}$ and $z = 3$. Define $X := X_A \cup \{z\}$. By construction, $X \in \mathcal{P}(A \cup B)$, But $X \notin \mathcal{P}(A)$ and $X \notin \mathcal{P}(B)$, therefore $X \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

**19.** Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. HOMEWORK.

**20.** Show that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$. HOMEWORK.

Exercises for Section 7.4

## Functions & Countability

**Theorem 1.** For any set $S$: $S$ has cardinality less (is 'smaller') than $\mathcal{P}(S)$.

**Proof.** (1) Since there is here is a one-to-one function from any set to its power set (the function that takes each element $a$ to the singleton set $\{a\}$), this implies that the cardinality of any set is less than or equal the cardinality of its power set. It is enough to prove that there is no surjective function from $\mathcal{P}(S)$ to $S$.

(2) By contradiction. Suppose $S$ has cardinality equal to or greather than $\mathcal{P}(S)$, that is, there is a surjective function, $f$, with:

$$f : S \longrightarrow \mathcal{P}(S)$$

(3) Define the following set:

$$A := \{x \in S \mid x \notin f(x)\} \subseteq S$$

By construction, $A \subseteq S$ and thus $A \in \mathcal{P}(S)$.

(4) Since $f$ is assumed to be surjective, there must be an $\bar{x} \in S$ such that $f(\bar{x}) = A$.

(5) Now we can distinguish the following two exhaustive cases:

**Case 1:** Assume $\bar{x} \in A$: Then by definition of $A$, we obtain $\bar{x} \notin f(\bar{x}) = A$, which is impossible.

**Case 2:** Assume $\bar{x} \notin A$: Then by choice of $\bar{x}$ with $f(\bar{x}) = A$, $\bar{x} \notin f(\bar{x})$, which by definition of $A$ implies $\bar{x} \in A$, which is impossible.

Both lead to contradiction, which proves the theorem.

**Bonus Question.** The odd and even integers have the same cardinality.

**Answer:** Define $f : \mathit{Even} \to \mathit{Odd}$ by setting $f(n) = n + 1$. Show that $f$ is a bijection.