Exercises for Discrete Maths

Discrete Maths

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Week 1

Computer Science

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Exercises for Section 4.1

Exercise 2a, 2b. Assume that m and n are particular integers. Is $6m+8n$ even? Is $10mn+7$ odd?

Turn the above questions into conjectures. If you believe them true, try proving them.
2a) $6m+8n = 2(3m+4n)$ which is even by definition.

2b) $10mn+7 = 10mn+6+1 = 2(5mn)+6+1 = 2(5mn+3)+1$ which is odd by definition.

Exercise 4 and 5, Existential proof. We are asked to prove that there exist $m, n \in \mathbb{Z}$ such that $\frac{1}{m} + \frac{1}{n} \in \mathbb{Z}$. Take: $m = 2$ and $n = 2$...

We are asked to prove that there exist $m, n \in \mathbb{Z}$ such that $m \neq n$ *and* $\frac{1}{m} + \frac{1}{n} \in \mathbb{Z}$. Take: $m = 3$ and $m = -3$. The result is 0.

Exercise 12, Counter-example. We have to disprove the following statement: $\forall n \in \mathbb{Z}$ *if* n is odd *then* $\frac{n-1}{2}$ is odd. In other words, we have to show what follows: $\exists n \in \mathbb{Z}$ that is odd *and* so that $\frac{n-1}{2}$ is *not* odd. Take $n = 5$...

Exercise 13, Counter-example. We have to disprove the following statement: $\forall m, n \in \mathbb{Z}$ *if* $2m+n$ is odd *then* m *and* n are odd. In other words, we have to show what follows: $\exists m, n \in \mathbb{Z}$ so that $2m+n$ is odd *and* so that m *or* n are *not* odd. Take $m = 4$ and $n = 1$...

Exercise 25, Universal proof. We have to prove the following statement: $\forall m, n \in \mathbb{Z}$ *if* m is even and n is odd *then* $m-n$ is odd. Let us spell out the definitions of being even and odd, respectively: $\exists k \in \mathbb{Z}$ so that $m = 2k$; $\exists r \in \mathbb{Z}$ so that $n = 2r+1$ (why can't we use k again?). Then

$$\begin{aligned} m - n &= 2k - (2r+1) = 2k - 2r - 1 = 2(k-r) - 1 \\ &= 2(k-r) + (-2+1) = (2(k-r)-2) + 1 = 2((k-r)-1) + 1 \end{aligned}$$

and, since $((k-r)-1) \in \mathbb{Z}$, then $2((k-r)-1)+1$ is odd by definition.

Exercise 29, Universal proof *(Homework)*. We have to prove the following statement: $\forall n \in \mathbb{Z}$ *if* n is odd *then* $3n+5$ is even. Let us spell out the definition of being odd: $\exists k \in \mathbb{Z}$ so that $n = 2k+1$. Therefore $3n+5 = 3(2k+1)+5 = 6k+3+5 = 6k+8 = 2(3k+4)$ which is even by definition.

Exercise 47. Consider the following statement: $\forall m, n \in \mathbb{Z}$ *if* $m+n$ is even *then* m or n are even. To show that the universal statement is false we need a counter-example. Take $m = n = 3$...

Exercises for Section 4.6

Exercise 3, Contradiction. We have to prove the following statement: $\forall n \in \mathbb{Z}$, 3 does not divide $3n+2$. We reason by contradiction and start our proof by negating the statement: $\exists n$ with $3n+2$ and $3|(3n+2)$ (a shorthand for "3 divides $(3n+2)$"). Then, by definition of $|$ (see Definition 1 in Section 4.3), there exists q with $3q = 3n+2$, hence $3q - 3n = 2$. This yields that $3(q-n) = 2$, which is false because $3 \not| 2$ (see Section 4.3, e.g., Theorem 4.3.1).

Exercise 17, Contradiction *(Homework)*. We have to prove the following statement: $\forall n \in \mathbb{Z}$ *if* $n \bmod 6 = 3$ *then* $n \bmod 3 \neq 2$. First, have a look at the Quotient Theorem 4.4.1. Then reason by contradiction and negate the statement, thereby obtaining: $\exists n \in \mathbb{Z}$ for which the if-then statement is false.

Exercise 19, Contraposition. We have to prove the following statement: $\forall n, m \in \mathbb{N}$ *if* $m.n > 100$ *then* $n > 10$ *or* $m > 10$. Let us reason by contraposition on the if-then statement: $\forall n, m \in \mathbb{N}$, *if* $n \not> 10$ *and* $m \not> 10$ *then* $m.n \not> 100$. This if-then statement is equivalent to the following: *if* $n \leq 10$ *and* $m \leq 10$ *then* $m.n \leq 100$. Let us assume that $n \leq 10$ *and* $m \leq 10$. Then, since m and n are non-negative, $m.n \leq 10.10 = 100$.

Exercise 28, Contraposition and Contradiction *(Homework)*. We are asked to prove the following statement first by contraposition and then by contradiction: $\forall n, m \in \mathbb{Z}$ *if* $m.n$ is even *then* m *or* n is even.

Let us start reasoning by contraposition on the if-then statement: $\forall m, n \in \mathbb{Z}$, *if* m *and* n are odd *then* $m.n$ is odd. Spelling out the definition of being odd, we have what follows: $\exists q, r$ with $m = 2q+1$ and $n = 2r+1$. Therefore $m.n = (2q+1).(2r+1) = 2.2rq + 2q + 2r + 1 = 2(2rq + q + r) + 1$, which is odd by definition.

Let us now reason by contradiction and assume that the given statement is false: $\exists m, n \in \mathbb{Z}$ so that $m.n$ is even *and* m *and* n are odd. Spelling out the definition of being even and odd, we have what follows: $\exists q$ with $m.n = 2q$; $\exists r$ with $m = 2r+1$; $\exists s$ with $n = 2s+1$. Therefore:

$$2q = m.n = (2r+1).(2s+1) = 2(2rs + r + s) + 1$$

which contradicts that an integer cannot be both even and odd.