Proving languages not to be regular

Consider: $L_{alt} = \{w \mid$ has alternating 0's and 1's $\}$

$L_{eq} = \{w \mid$ has an equal number of 0's and 1's $\}$

- Claim: L_{alt} is regular

  Proof: easy $E_{alt} = (\varepsilon + 0)(1 \cdot 0)^* \cdot (\varepsilon + 1)$ is such that $\mathcal{L}(E_{alt}) = L_{alt}$

- Claim: L_{eq} is not regular

  How can we prove this?

  Intuition: · DFA with n states can count up to n.

  · to decide whether $w \in L_{eq}$ we need unbounded counting (since w may be arbitrarily long)

Pumping Lemma:

For all regular languages $L \subseteq \Sigma^*$

there exists n (which depends on L) such that

for all $w \in L$ with $|w| \geq n$

there exists a decomposition $w = xyz$ of w s.t.

1) $|y| \geq 1$ (i.e., $y \neq \varepsilon$)
2) $|x \cdot y| \leq n$
3) for all $k \geq 0$, $x y^k z \in L$.

Intuitively, for every $w \in L$, we can find a substring y "near" the beginning of w that can be "pumped", while still obtaining words in L.

OPTIONAL

Proof:

Given regular language L, let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA with $\mathcal{L}(A) = L$.

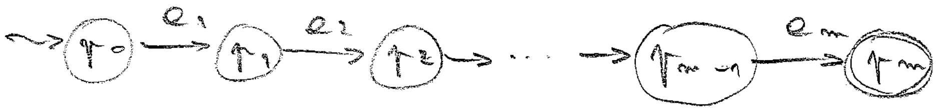
We take $n = |Q|$.

Consider any $w = a_1 a_2 \cdots a_m \in L$ with $m = |w| \geq n$.

Since $w \in \mathcal{L}(A)$, we have that $\hat{\delta}(q_0, w) \in F$.

Define $p_i = \hat{\delta}(q_0, a_1 a_2 \cdots a_i)$ $\forall i \in \{1, \dots, m\}$ and

$p_0 = q_0$



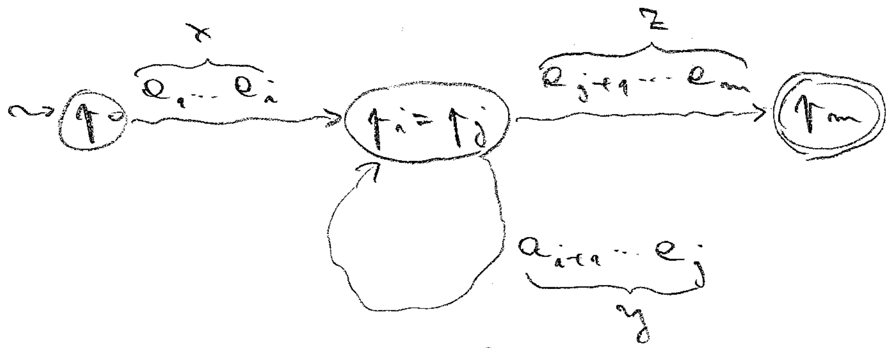
Since $m \geq n$,

- each p_i, $0 \leq i \leq m$ belongs to Q, and
- $|Q| = n$

by the pigeon-hole principle, $p_0, p_1, \dots, p_m$ are not all distinct.

Let i, j, with $0 \leq i < j \leq n$ be the least indices such that $p_i = p_j$.

Hence, to accept w, the DFA goes through a cycle:



$\hat{\delta}(p_0, x) = p_i$

$\hat{\delta}(p_i, y) = p_i$

$\hat{\delta}(p_i, z) = p_m$

Observe: $|y| = j - i \geq 1$ (since $i < j$)

- $|xy| = j \leq n$

$\hat{\delta}(q_0, xy^k z) = \hat{\delta}(\hat{\delta}(q_0, x), y^k z) = \hat{\delta}(p_i, y^k z) = \hat{\delta}(\hat{\delta}(p_i, y), y^{k-1} z)$

$= \hat{\delta}(p_i, y^{k-1} z) = \cdots = \hat{\delta}(p_i, z) = p_m \in F \Rightarrow xy^k z \in L$

q.e.d.

OPTIONAL END