Algorithms for Data Processing
Lecture V: Network Flow

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Alberto Montresor, Werner Nutt, Kevin Wayne, Jon Kleinberg, Eva Tardos.

Flow Networks

- Use of graphs to model transportation networks: networks whose edges carry some sort of traffic and whose nodes act as switches passing traffic between different edges.
  - Example. Fluid network in which edges are pipes that carry liquid, and the nodes are junctures where pipes are plugged together.

Flow Networks – Ingredients

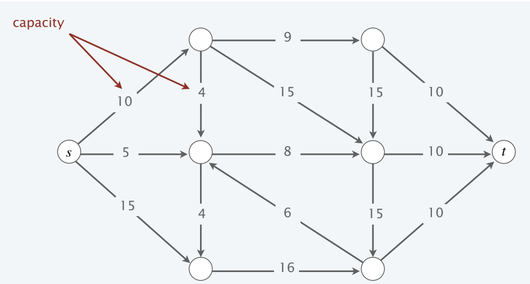
- Capacity on the edge, indicating how much “traffic” can carry;
- Source nodes in the graph, which generate traffic;
- sink (or destination) nodes in the graph, which can "absorb" traffic as it arrives;
- traffic or flow which is currently transmitted across an edge.

Flow Networks – Ingredients/2

A Flow Network is a tuple $G = (V, E, s, t, c)$, where:

- (V, E) is a directed graph, with
- a single source $s \in V$, and a single sink $t \in V$;
- Nodes other than s and t will be called internal nodes.
- Capacity $c(e) > 0$ for each $e \in E$.

Intuition. Material flowing through a transportation network, originating at source and sent to sink.



Flow Networks – Ingredients/3

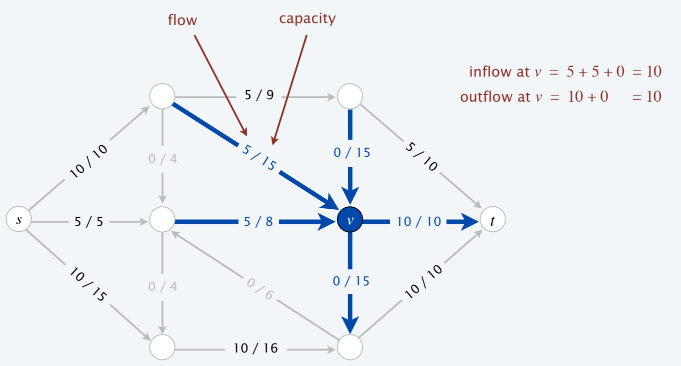
We make the following assumptions:

- No edge enters the source s, and no edge leaves the sink t;
- There is at least one edge incident to each node;
- Capacities are integers.

Flow Networks – Ingredients/4

We now define what it means for our network to carry traffic or flow.

- An *st*-flow is a function, $f : E \to \mathbf{R}^+$, that satisfies:
  Capacity Condition $\forall e \in E. 0 \leq f(e) \leq c(e)$
  Conservation Condition $\forall v \in V \setminus \{s, t\}. \sum_{e \ into \ v} f(e) = \sum_{e \ out \ of \ v} f(e)$

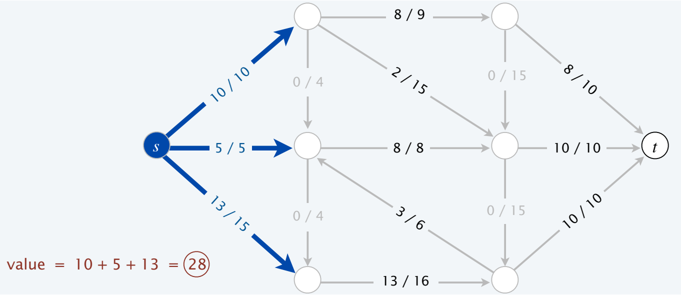


Flow Networks – Ingredients/5

Value of a flow. Amount of flow generated at the source:

$$val(f) = \sum_{e \text{ out of } s} f(e)$$

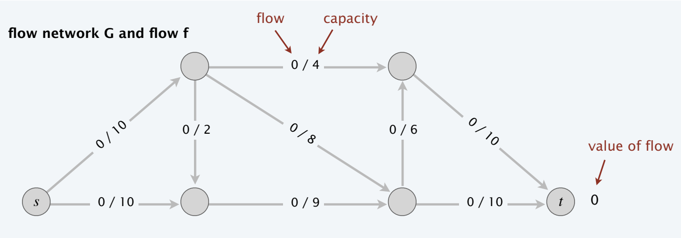
Max-Flow Problem. Find a flow of maximum value.



Toward a Max-Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an s-t path P where each edge has $f(e) < c(e)$.
- Augment flow along path P.
- Repeat until you get stuck!



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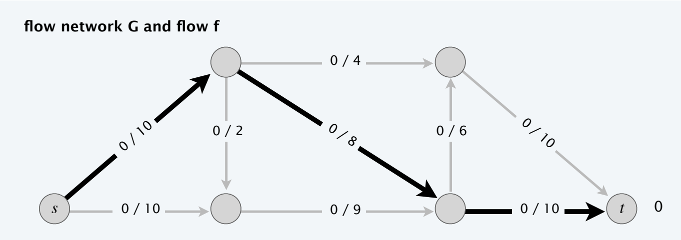

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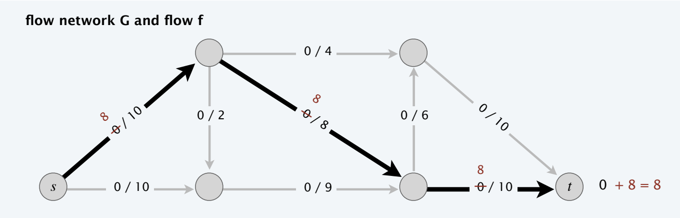

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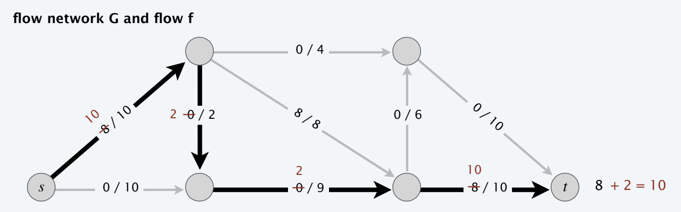

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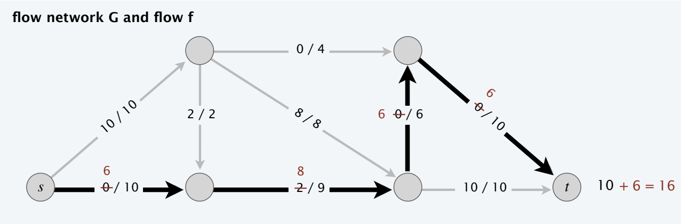

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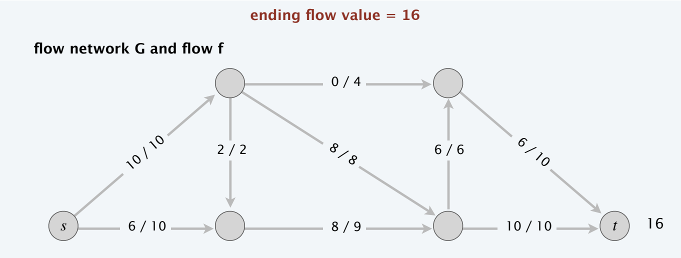

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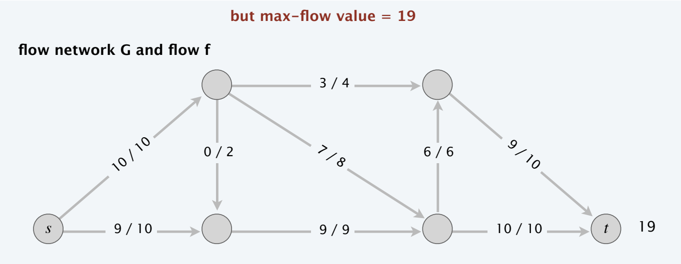


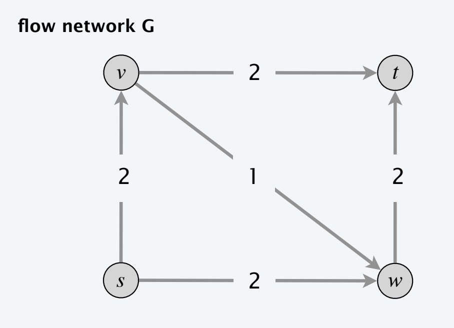
Why the greedy algorithm fails

Q. Why does the greedy algorithm fail?

A. Greedy algorithm never decreases the flow on an edge.

Example. Consider the flow network G below.

- The unique max flow has $f(v, w) = 0$.
- Greedy algorithm could choose $s \to v \to w \to t$ as first augmenting path.



Mechanism to Undo a Bad Decision

This is a more general way of pushing flow:

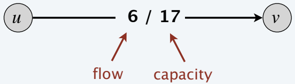
- We can push forward on edges with leftover capacity;
- We can push backward on edges that are already carrying flow, to divert it in a different direction.

We now define the residual graph, which provides a systematic way to search for *forward-backward* operations.

Residual network

Original edge. $e = (u, v) \in E$.

- Flow $f(e)$.
- Capacity $c(e)$.

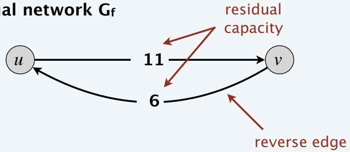


Reverse edge. $e^{\text{reverse}} = (v, u)$.

- "Undo" flow sent.

Residual capacity.

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^{\text{reverse}} \in E \end{cases}$$



edges with positive residual capacity

where flow on a reverse edge negates flow on corresponding forward edge

Residual network. $G_f = (V, E_f, s, t, c_f)$.

- $E_f = \{e : f(e) < c(e)\} \cup \{e^{\text{reverse}} : f(e) > 0\}$.
- Key property: f' is a flow in G_f iff $f + f'$ is a flow in G.

Augmenting Path

Augmenting Path. A simple $s - t$ path in the *residual network* G_f, for a given graph G and a flow f.

Bottleneck Capacity. Minimal residual capacity of any edge in an augmenting path.

Note. In the following a *reverse* edge will be denoted as a backward edge.

Augmenting Path/2

The following algorithm makes precise the way in which we push flow from s to t in G as a consequence of an augmenting path, P, in the residual network G_f.

```
AUGMENT(f,P)
b = BOTTLENECK(P, f);
for each edge (u, v) ∈ P do
    if e = (u, v) is a forward edge then
        f(e) = f(e) + b in G
    else
        f(e) = f(e) − b in G;        /* (u, v) is a backward edge, and e = (v, u) */
return f
```

Key Property. The result of `AUGMENT(f,P)` is a new flow f' in G such that $val(f') = val(f) + b$.

Ford-Fulkerson Algorithm (1956)

```
MAX-FLOW(G)
f(e) = 0 for all e ∈ G;
G_0 = G;                                  /* Initialize the residual graph G_0 */
while there is an s-t path, P, in the residual graph G_f do
    if P is a simple s-t path in G_f then
        f' = AUGMENT(f, P);
        G_f' = UPDATE(G_f,P,f');                        /* Update G_f to G_f' */
        f ← f';
return f
```

```
UPDATE(G_f,P,f')
compute the bottleneck b from f and f';
for each (u, v) in P do
    c(u,v) = c(u,v) - b;
    if c(u,v) =0 then
        delete edge (u, v) from G_f
    if (v, u) ∉ G_f then
        add edge (v, u) to G_f
    c(v,u) = c(v,u) + b;
return G_f
```

Correctness of the Ford-Fulkerson Algorithm

Property/1. Let f be a flow in G, and let P be a simple s-t path in G_f. Then, $val(f') = val(f) + \texttt{bottleneck}(P, f)$; and since $\texttt{bottleneck}(P, f) > 0$, we have $val(f') > val(f)$.

Property/2. Let $C = \texttt{max}_e\{c(e)\}$, then, the Ford-Fulkerson Algorithm terminates in at most nC iterations of the `While loop`.

Proof. Note that:

- $val(f_{max}) \leq \sum_{e \text{ out of } s} c(e) \leq nC$;
- By Property/1, the value of the flow increases at each iteration by at least 1 unit.

Note! The book has a different choice: $C = \sum_{e \text{ out of } s} c(e)$

Running Time of the Ford-Fulkerson Algorithm

Assuming that all nodes have at least one incident edge, then $m > n/2$, and so we can say that $O(m+n) = O(m)$.

Running Time. The Ford-Fulkerson Algorithm can be implemented to run in $O(mnC)$ time.

Proof. Complexity in one iteration of the `While` loop.

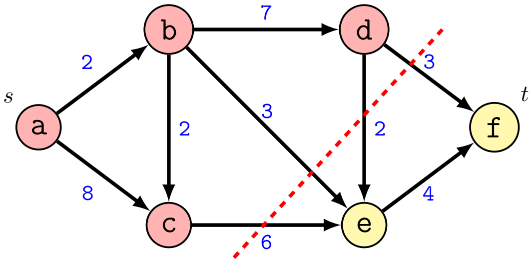
- The residual graph G_f has at most $2m$ edges;
- G_f is stored using an adjacency list;
- To find an s-t path in G_f, we can use BFS or DFS, which runs in $O(m+n)$ time which, by our assumption, is the same as $O(m)$;
- The procedure `AUGMENT`(f, P) takes time $O(n)$, as the path P has at most $n-1$ edges;
- The procedure `UPDATE`(G_f, P, b) takes also time $O(n)$.

Insight in the Max-Flow Problem

- We continue with the analysis of the Ford-Fulkerson Algorithm.
- Objective: to find considerable insights into the Maximum-Flow problem itself.
- We already saw that $val(f) \leq \sum_{e \text{ out of } s} c(e)$.
- Is there a better approximation of $val(f)$?

Cut in a Flow Network

Cut in a Flow Network. A cut, (S, T), in a Flow Network $G = (V, E, s, t, c)$ is called an s–t cut if it is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.

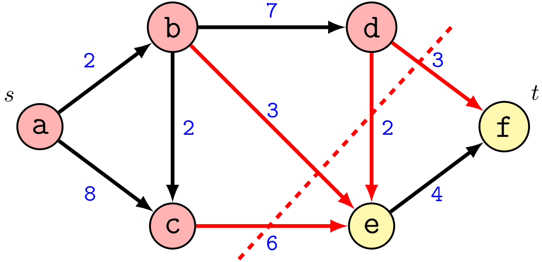


$S = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$
$T = \{\mathtt{e}, \mathtt{f}\}$

Cut in a Flow Network/2

Capacity across the cut. The capacity across a cut, $c(S, T)$, is given by the following formula:

$$c(S, T) = \sum_{u \in S, v \in T} c(u, v) = \sum_{e \text{ out of } S} c(e)$$



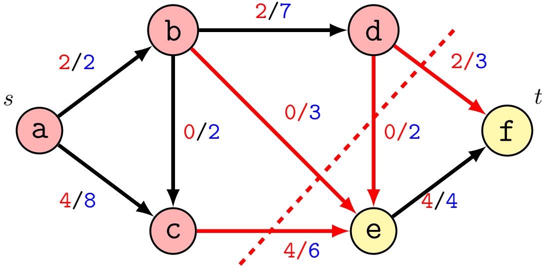
$S = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$
$T = \{\mathtt{e}, \mathtt{f}\}$

$C(S, T) = 14$

Cut in a Flow Network/3

Flow across the cut. The flow across a cut, $f(S,T)$, is given by the following formula:

$$f(S,T) = \sum_{u \in S, v \in T} f(u,v) - \sum_{v \in T, u \in S} f(v,u) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e)$$



$S = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$
$T = \{\mathtt{e}, \mathtt{f}\}$

$C(S,T) = 14$

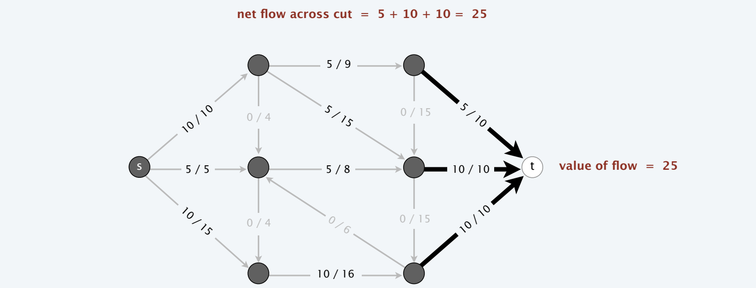
$F(S,T) = 6$

Flow value Lemma

Cuts provide very natural upper bounds on the values of flows.

Flow value Lemma. Let f be any s–t flow and (S, T) be any s–t cut. Then, the value of the flow f equals the flow across the cut (S, T):

$$val(f) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e)$$



Flow value Lemma/2

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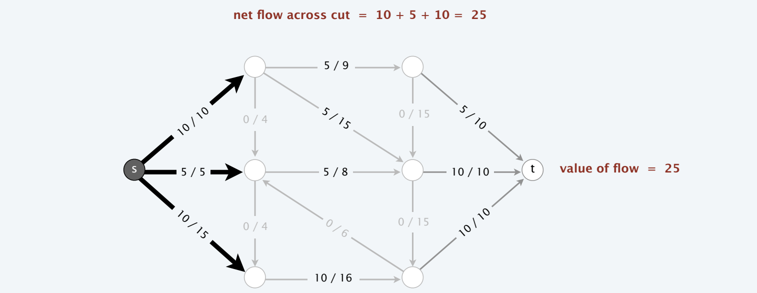

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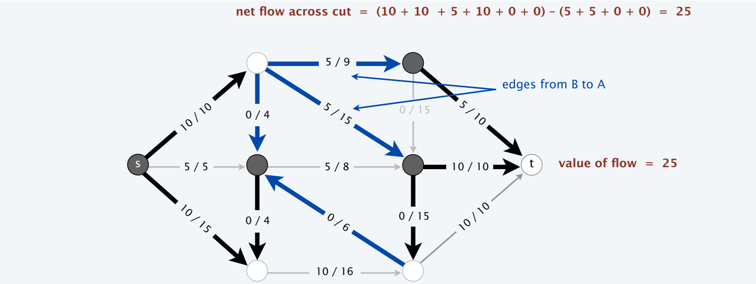


Flow value Lemma – Proof

Flow value Lemma. Let f be any s–t flow and (S, T) be any s–t cut. Then, the value of the flow f equals the flow across the cut (S, T):

$$val(f) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e)$$

Proof.

$$val(f) = \sum_{e \text{ out of } s} f(e) = \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ into } s} f(e) =$$

$$= \sum_{v \in S} \Big(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \Big) \; [\textit{all are 0 except for } v{=}s]$$

$$= \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e)$$

Relationship between Flow and Capacity

Weak duality Lemma. Let f be any s–t flow, and (S, T) any s–t cut. Then,

$$val(f) \leq c(S, T).$$

Proof.

$$\begin{aligned} val(f) &= \sum_{e \ out \ of \ S} f(e) - \sum_{e \ into \ S} f(e) \leq \\ &\leq \sum_{e \ out \ of \ S} f(e) \leq \\ &\leq \sum_{e \ out \ of \ S} c(e) = c(S, T). \end{aligned}$$

Relationship between Flow and Capacity/2

- The Weak duality Lemma abstracts from a partcular flow;
- The value of every flow is upper-bounded by the capacity of any possible cut;
- Thus, if we exhibit an s-t cut in G of some value $c(S, T)$ we know that there cannot be an s-t flow in G of value greater than $c(S, T)$.
- Viceversa, if we exhibit any s-t flow, f^* in G, we know that there cannot be an s-t cut in G with s-t capacity less than f^*.

Certificate of optimality

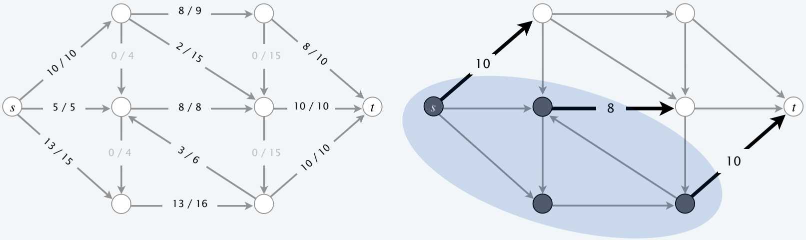
Corollary. Let f be a flow and let (A, B) be any cut.
If $val(f) = cap(A, B)$, then f is a max flow and (A, B) is a min cut.

Pf.

weak duality

- For any flow f': $val(f') \leq cap(A, B) = val(f)$.
- For any cut (A', B'): $cap(A', B') \geq val(f) = cap(A, B)$. ▪

weak duality



value of flow = 28 = capacity of cut = 28

Max-flow min-cut theorem

Max-flow min-cut theorem. Value of a max flow = capacity of a min cut.

strong duality

MAXIMAL FLOW THROUGH A NETWORK

L. R. FORD, JR. AND D. R. FULKERSON

Introduction. The problem discussed in this paper was formulated by T. Harris as follows:

"Consider a rail network connecting two cities by way of a number of intermediate cities, where each link of the network has a number assigned to it representing its capacity. Assuming a steady state condition, find a maximal flow from one given city to the other."

ON THE MAX FLOW MIN CUT THEOREM OF NETWORKS

G. B. Dantzig
D. R. Fulkerson

P-826

April 15, 1955

A Note on the Maximum Flow Through a Network*

P. ELIAS†, A. FEINSTEIN‡, AND C. E. SHANNON§

Summary—This note discusses the problem of maximizing the rate of flow from one terminal to another, through a network which consists of a number of branches, each of which has a limited capacity. The main result is a theorem: The maximum possible flow from left to right through a network is equal to the minimum value among all simple cut-sets. This theorem is applied to solve a more general problem, in which a number of input nodes and a number of output nodes are used.

from one terminal to the other in the original network passes through at least one branch in the cut-set. In the network above, some examples of cut-sets are (d, e, f), and (b, c, e, g, h), (d, g, h, i). By a *simple cut-set* we will mean a cut-set such that if any branch is omitted it is no longer a cut-set. Thus (d, e, f) and (b, c, e, g, h) are simple cut-sets while (d, g, h, i) is not. When a simple cut-set is

Analyzing the Ford-Fulkerson Algorithm

- Let $\overline{f}$ denote the flow that is returned by the Ford-Fulkerson Algorithm. We want to prove that $\overline{f}$ is the maximum-flow by showing that there is an (S, T) cut such that:

$$val(\overline{f}) = c(S, T)$$

- The Ford-Fulkerson Algorithm terminates when for the flow f there is no $s - t$ path in the residual graph G_f.
  This is the only property needed for proving its maximality.

Correctness the Ford-Fulkerson Algorithm/1

Theorem 1. If f is an s-t flow such that there is no s-t path in the residual graph G_f (Augmenting Path) then there is an s-t cut (A, B) in G for which $val(f) = c(A, B)$.

Proof.

- Let A be set of nodes reachable from s in the residual network G_f and $B = V \setminus A$. We show that (A, B) is an s-t cut:
  - (A, B) is a partition of V, and $s \in A$, and $t \notin A$ by the assumption that there is no s-t path in G_f, hence $t \in B$.
- Let $e = (u, v)$ be an edge in G for which $u \in A$ and $v \in B$, then $f(e) = c(e)$ – otherwise e would be a forward edge in G_f.
- Let $e' = (u', v')$ be an edge in G for which $u' \in B$ and $v' \in A$, then $f(e') = 0$ – otherwise e' would give rise to a backward edge $e'' = (v', u')$ in G_f.
- So all edges out of A are completely saturated with flow, while all edges into A are completely unused.

Correctness the Ford-Fulkerson Algorithm/2

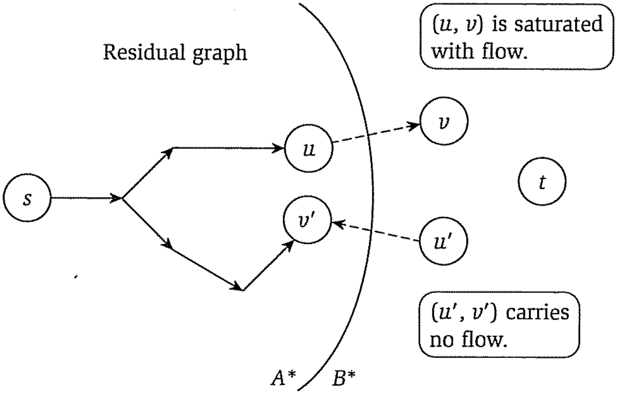


Figure 7.5 The (A^*, B^*) cut in the proof of (7.9).

Correctness the Ford-Fulkerson Algorithm/3

- We can now use the Flow value Lemma:

$$
\begin{aligned}
val(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = \\
&= \sum_{e \text{ out of } A} c(e) - 0 \\
&= c(A, B)
\end{aligned}
$$

That proves the Theorem!

Theorem 2. The flow $\overline{f}$ returned by the Ford-Fulkerson Algorithm is a maximum flow.

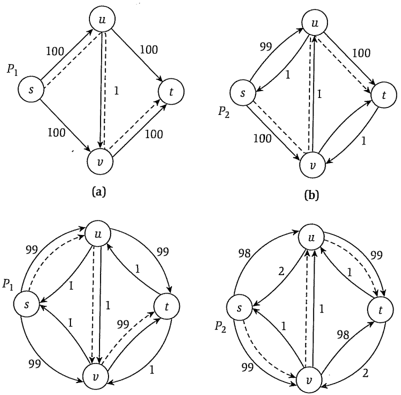
Integer-Value Flows

When all capacities are integer values we can guarantee the existence of a max-flow as expressed in the following Theorem.

Theorem [Integrality theorem.] If all capacities in the flow network are integers, then there is a maximum flow f for which every flow value $f(e)$ is an integer.

Choosing Good Augmenting Paths

- We already saw that $val(f) \leq \sum_{e \text{ out of } s} c(e)$ is an upper bound to the number of iterations.
- The Ford-Fulkerson Algorithm can perform vary badly when pathological augmenting paths are selected: Here we need 200 steps!!



Choosing Good Augmenting Paths

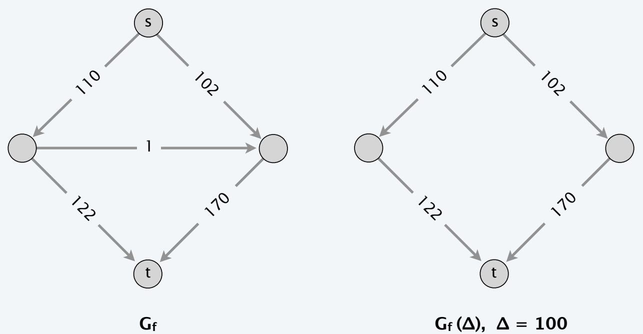
Use care when selecting augmenting paths.

- If we choose paths with large bottleneck capacity we will require less iterations.
- A natural idea is to select at each iteration the path that has the largest bottleneck capacity.
- Finding such paths can slow down each iteration.

Capacity-Scaling Algorithm

Main Idea: Choosing augmenting paths with large bottleneck capacity though not necessarily the largest.

- Maintain a scaling parameter Δ;
- Let $G_f(\Delta)$ be the sub-graph of the residual network G_f containing only those edges with residual capacity $\geq \Delta$;
- Any augmenting path in $G_f(\Delta)$ has bottleneck capacity $\geq \Delta$.



Capacity-Scaling Algorithm/2

```
CAPACITY-SCALING(G,s,t)
f(e) = 0 for all e ∈ G;
Δ = largest power of 2 ≤ max_{e out of s}{c(e)};
Compute G_0(Δ);
while Δ ≥ 1 do
    while there is an s-t path, P, in the residual graph G_f(Δ) do
        if P is a simple s-t path then
            f' = AUGMENT(f, P);
            G_f'(Δ) = UPDATE(G_f(Δ),P,f');          /* Update G_f(Δ) to G_f'(Δ) */
            f ← f';
    Δ = Δ/2
return f
```

Correctness of the Capacity-Scaling Algorithm

- The Capacity-Scaling Max-Flow Algorithm is just an optimized implementation of the original Ford-Fulkerson Algorithm.
- The search in the restricted residual graph $G_f(\Delta)$ is used to guide the selection of augmenting paths with large residual capacity.

Properties of Capacity-Scaling Algorithm. If the capacities are integer-valued, then throughout the Capacity-Scaling Max-Flow algorithm the flow and the residual capacities remain integer-valued. This implies that when $\Delta = 1$, $G_f(\Delta)$ is the same as G_f and hence when the algorithm terminates the flow, f, is of maximum value.

Capacity-Scaling Algorithm: Run Time Analysis

- We call an iteration of the outside `While` loop, with a fixed value of Δ, the Δ-scaling phase.
- We denote $C = \mathtt{max}_e\{c(e)\}$.

Lemma 1. There are $1 + \lfloor \log_2 C \rfloor$ Δ-scaling phases.
Proof. Initially $C/2 < \Delta \leq C$; Δ decreases by a factor of 2 in each iteration.

Lemma 2. During the Δ-scaling phase each augmentation increases the flow value by at least Δ.
Proof. During the Δ-scaling phase we only use edges with residual capacity of at least Δ.

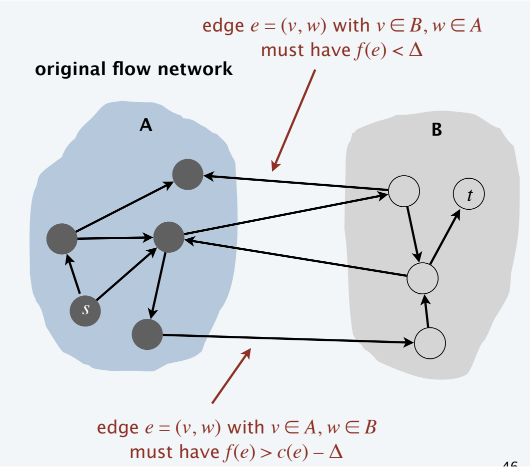
Capacity-Scaling Algorithm: Run Time Analysis/2// (Proof Not Required)

Lemma 3. Let f be the flow at the end of a Δ-scaling phase. Then, the max-flow value $\leq val(f) + m\Delta$ (where m is the number of edges).

Proof.

- We show there exists a cut (A, B) such that $c(A, B) \leq val(f) + m\Delta$.
- Let A be the set of nodes reachable from s in $G_f(\Delta)$ and $B = V \setminus A$. We show that (A, B) is an s–t cut:
  - (A, B) is a partition of V, and $s \in A$, and $t \notin A$ for otherwise there is an s–t path in $G_f(\Delta)$, hence $t \in B$.
- Let $e = (u, v)$ be an edge in G for which $u \in A$ and $v \in B$, then $c(e) - f(e) < \Delta$, otherwise e would be a forward edge in $G_f(\Delta)$, contradicting $v \in B$.
- Let $e' = (u', v')$ be an edge in G for which $u' \in B$ and $v' \in A$, then $f(e') < \Delta$, otherwise e' would give rise to a backward edge $e'' = (v', u')$ in $G_f(\Delta)$.
- So all edges out of A are almost saturated ($f(e) > c(e) - \Delta$), while all edges into A are almost empty ($f(e) < \Delta$).

Capacity-Scaling Algorithm: Run Time Analysis/3 (Proof Not Required)



Capacity-Scaling Algorithm: Run Time Analysis/4 (Proof Not Required)

Proof of Lemma 3 (cont.)

$$\begin{aligned} val(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \geq \quad [\texttt{By the Flow Value Lemma}] \\ &\geq \sum_{e \text{ out of } A} \big(c(e) - \Delta\big) - \sum_{e \text{ into } A} \Delta = \\ &= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ into } A} \Delta \geq c(A, B) - m\Delta. \end{aligned}$$

Thus, $c(A, B) \leq val(f) + m\Delta$, which proves Lemma 3.

Capacity-Scaling Algorithm: Run Time Analysis/5 (Proof Not Required)

Lemma 4. The number of augmentations in each scaling phase is $\leq 2m$.
Proof.

- True in the first scaling phase: we can have as many augmenting paths as many edges out of s (note that by Lemma 2, each augmenting phase increases the flow by at least Δ);
- In any later Δ-scaling phase, let f_p the flow at the end of the *previous* scaling phase;
- In the *previous* scaling phase we had $\Delta_p = 2\Delta$;
- By Lemma 3, $val(f_{max}) \leq val(f_p) + m\Delta_p = val(f_p) + 2m\Delta$;
- In the current Δ-scaling phase, by Lemma 2, each augmentation increases the flow by at least Δ, and hence there can be at most $2m$ augmentations.

Capacity-Scaling Algorithm: Run Time Analysis/6 (Proof Not Required)

Theorem. The Scaling Max-Flow algorithm in a graph with m edges and integer capacities finds a maximum flow in at most $2m(1 + \lfloor \log_2 C \rfloor)$ augmentations. It can be implemented to run in at most $O(m^2 \log_2 C)$ time.

Proof.

- By Lemmas 1 and 4, we can have at most $2m(1 + \lfloor \log_2 C \rfloor)$ augmentations, i.e., $O(m \log_2 C)$;
- Each augmentation takes $O(m)$ including the time to find a path (BFS/DFS) and to generate the new residual graph.

Considerations on Running Time

- When C is large, the scaling algorithm, $O(m^2 \log_2 C)$, outperforms the generic implementation of the Ford-Fulkerson Algorithm, $O(mnC)$.

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- When the generic Ford-Fulkerson algorithm chooses patological paths could require C iterations, i.e., exponential time in the size of the bit representation of the input.
- The scaling algorithm is running in time polynomial in the size of the input, i.e., the number of edges and the bit representation of the capacities.

Augmenting Path Algorithms: Summary

year	method	# augmentations	running time
1955	**augmenting path**	$n\,C$	$O(m\,n\,C)$
1972	**fattest path**	$m \log(mC)$	$O(m^2 \log n \log(mC))$
1972	**capacity scaling**	$m \log C$	$O(m^2 \log C)$
1985	**improved capacity scaling**	$m \log C$	$O(m\,n \log C)$
1970	**shortest augmenting path**	$m\,n$	$O(m^2\,n)$
1970	**level graph**	$m\,n$	$O(m\,n^2)$
1983	**dynamic trees**	$m\,n$	$O(m\,n \log n)$

fat paths

shortest paths

augmenting-path algorithms with m edges, n nodes, and integer capacities between 1 and C

Max-Flow Algorithms: Summary

year	method	worst case	discovered by
1951	**simplex**	$O(m\, n^2\, C)$	Dantzig
1955	**augmenting paths**	$O(m\, n\, C)$	Ford–Fulkerson
1970	**shortest augmenting paths**	$O(m\, n^2)$	Edmonds–Karp, Dinitz
1974	**blocking flows**	$O(n^3)$	Karzanov
1983	**dynamic trees**	$O(m\, n \log n)$	Sleator–Tarjan
1985	**improved capacity scaling**	$O(m\, n \log C)$	Gabow
1988	**push-relabel**	$O(m\, n \log (n^2 / m))$	Goldberg–Tarjan
1998	**binary blocking flows**	$O(m^{3/2} \log (n^2 / m) \log C)$	Goldberg–Rao
2013	**compact networks**	$O(m\, n)$	Orlin
2014	**interior-point methods**	$\tilde{O}(m\, m^{1/2} \log C)$	Lee–Sidford
2016	**electrical flows**	$\tilde{O}(m^{10/7}\, C^{1/7})$	Mądry
20xx		???	

max-flow algorithms with m edges, n nodes, and integer capacities between 1 and C

Thank You!