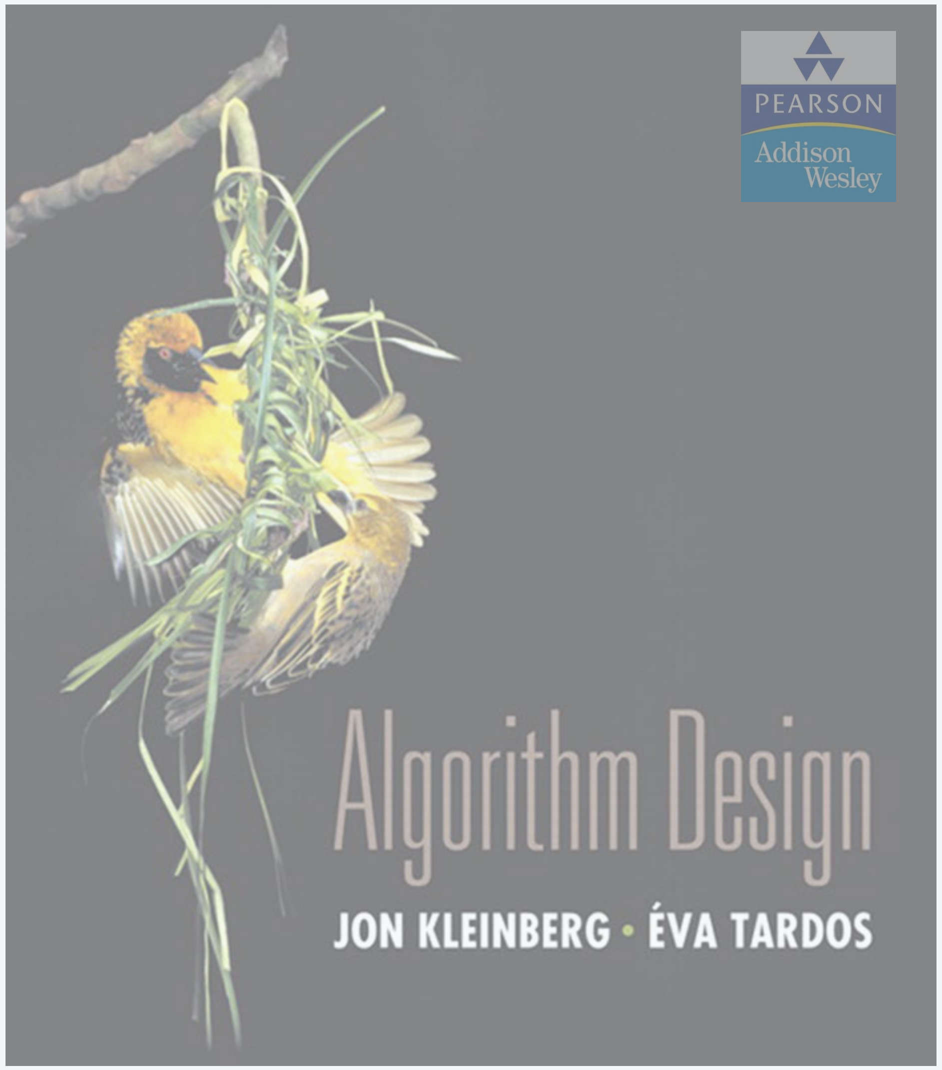


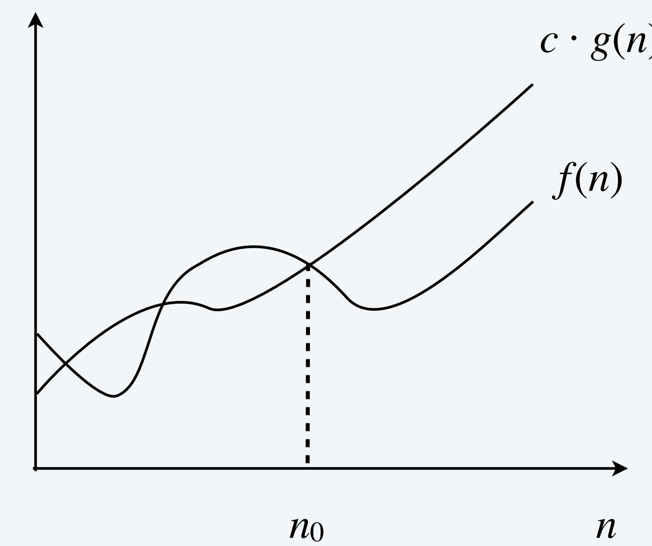
SECTION 2.2

2. ALGORITHM ANALYSIS

- *computational tractability*
- *asymptotic order of growth*
- *implementing Gale–Shapley*
- *survey of common running times*

Big O notation

Upper bounds. $f(n)$ is $O(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $0 \leq f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.



Ex. $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is $O(n^2)$. ⟵ choose $c = 50$, $n_0 = 1$
- $f(n)$ is neither $O(n)$ nor $O(n \log n)$.

Typical usage. Insertion sort makes $O(n^2)$ compares to sort n elements.

Analysis of algorithms: quiz 1

Let $f(n) = 3n^2 + 17\, n \log_2 n + 1000$. Which of the following are true?

A. $f(n)$ is $O(n^2)$.

B. $f(n)$ is $O(n^3)$.

C. Both A and B.

D. Neither A nor B.

Big O notation: properties

Reflexivity. f is $O(f)$.

Constants. If f is $O(g)$ and $c > 0$, then $c\,f$ is $O(g)$.

Products. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 f_2$ is $O(g_1 g_2)$.

Pf.

- $\exists\ c_1 > 0$ and $n_1 \geq 0$ such that $0 \leq f_1(n) \leq c_1 \cdot g_1(n)$ for all $n \geq n_1$.
- $\exists\ c_2 > 0$ and $n_2 \geq 0$ such that $0 \leq f_2(n) \leq c_2 \cdot g_2(n)$ for all $n \geq n_2$.
- Then, $0 \leq f_1(n) \cdot f_2(n) \leq \underbrace{c_1 \cdot c_2}_{c} \cdot g_1(n) \cdot g_2(n)$ for all $n \geq \underbrace{\max \{ n_1, n_2 \}}_{n_0}$. ▪

Sums. If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 + f_2$ is $O(\max \{g_1, g_2\})$.

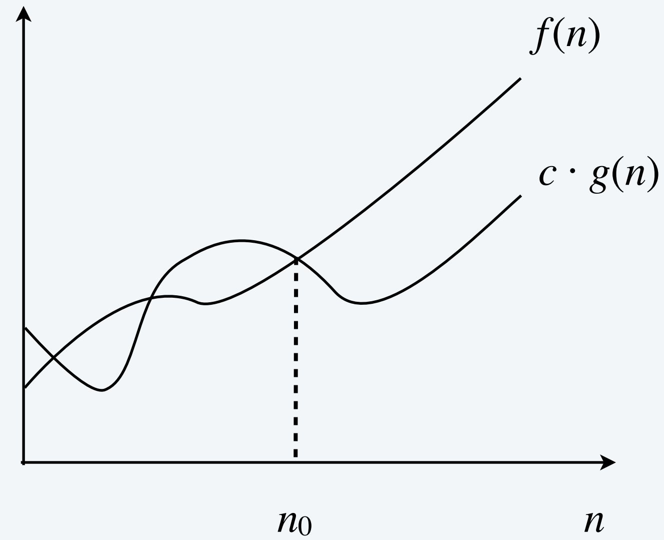
ignore lower-order terms

Transitivity. If f is $O(g)$ and g is $O(h)$, then f is $O(h)$.

Ex. $f(n) = 5n^3 + 3n^2 + n + 1234$ is $O(n^3)$.

Big Omega notation

Lower bounds. $f(n)$ is $\Omega(g(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that $f(n) \geq c \cdot g(n) \geq 0$ for all $n \geq n_0$.



Ex. $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is both $\Omega(n^2)$ and $\Omega(n)$. ⟵ choose $c = 32$, $n_0 = 1$
- $f(n)$ is not $\Omega(n^3)$.

Typical usage. Any compare-based sorting algorithm requires $\Omega(n \log n)$ compares in the worst case.

Vacuous statement. Any compare-based sorting algorithm requires at least $O(n \log n)$ compares in the worst case.

Which is an equivalent definition of big Omega notation?

A. $f(n)$ is $\Omega(g(n))$ iff $g(n)$ is $O(f(n))$.

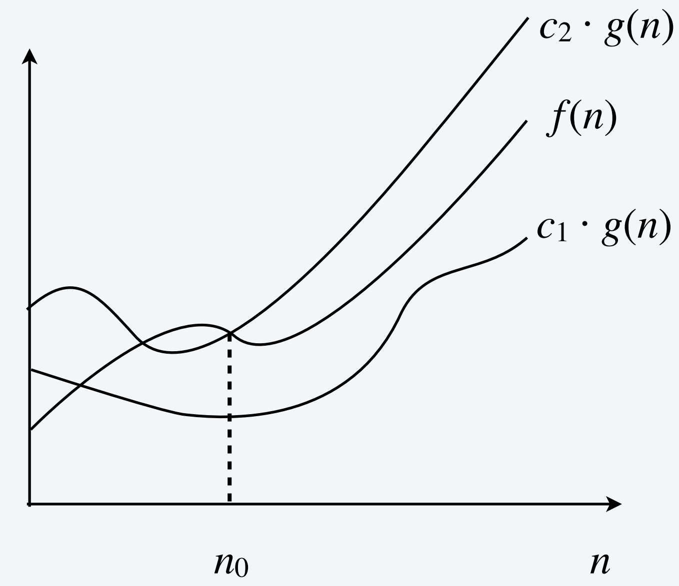
B. $f(n)$ is $\Omega(g(n))$ iff there exist constants $c > 0$ such that $f(n) \geq c \cdot g(n) \geq 0$ for infinitely many n.

C. Both A and B.

D. Neither A nor B.

Big Theta notation

Tight bounds. $f(n)$ is $\Theta(g(n))$ if there exist constants $c_1 > 0$, $c_2 > 0$, and $n_0 \geq 0$ such that $0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.



Ex. $f(n) = 32n^2 + 17n + 1$.

- $f(n)$ is $\Theta(n^2)$. ← choose $c_1 = 32$, $c_2 = 50$, $n_0 = 1$
- $f(n)$ is neither $\Theta(n)$ nor $\Theta(n^3)$.

Typical usage. Mergesort makes $\Theta(n \log n)$ compares to sort n elements.

↑ between $\frac{1}{2}\, n \log_2 n$ and $n \log_2 n$

Analysis of algorithms: quiz 3

Which is an equivalent definition of big Theta notation?

A. $f(n)$ is $\Theta(g(n))$ iff $f(n)$ is both $O(g(n))$ and $\Omega(g(n))$.

B. $f(n)$ is $\Theta(g(n))$ iff $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$ for some constant $0 < c < \infty$.

C. Both A and B.

D. Neither A nor B.

Asymptotic bounds and limits

Proposition. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c$ for some constant $0 < c < \infty$ then $f(n)$ is $\Theta(g(n))$.

Pf.

- By definition of the limit, for any $\varepsilon > 0$, there exists n_0 such that

$$c - \epsilon \leq \frac{f(n)}{g(n)} \leq c + \epsilon$$

  for all $n \geq n_0$.
- Choose $\varepsilon = ½\, c > 0$.
- Multiplying by $g(n)$ yields $1/2\, c \cdot g(n) \leq f(n) \leq 3/2\, c \cdot g(n)$ for all $n \geq n_0$.
- Thus, $f(n)$ is $\Theta(g(n))$ by definition, with $c_1 = 1/2\, c$ and $c_2 = 3/2\, c$. ▪

Proposition. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$, then $f(n)$ is $O(g(n))$ but not $\Omega(g(n))$.

Proposition. If $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$, then $f(n)$ is $\Omega(g(n))$ but not $O(g(n))$.

Asymptotic bounds for some common functions

Polynomials. Let $f(n) = a_0 + a_1 n + \ldots + a_d n^d$ with $a_d > 0$. Then, $f(n)$ is $\Theta(n^d)$.

Pf.

$$\lim_{n \to \infty} \frac{a_0 + a_1 n + \ldots + a_d n^d}{n^d} = a_d > 0$$

Logarithms. $\log_a n$ is $\Theta(\log_b n)$ for every $a > 1$ and every $b > 1$.

Pf. $$\frac{\log_a n}{\log_b n} = \frac{1}{\log_b a}$$

no need to specify base
(assuming it is a constant)

Logarithms and polynomials. $\log_a n$ is $O(n^d)$ for every $a > 1$ and every $d > 0$.

Pf.

$$\lim_{n \to \infty} \frac{\log_a n}{n^d} = 0$$

Exponentials and polynomials. n^d is $O(r^n)$ for every $r > 1$ and every $d > 0$.

Pf.

$$\lim_{n \to \infty} \frac{n^d}{r^n} = 0$$

Factorials. $n!$ is $2^{\Theta(n \log n)}$.

Pf. Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

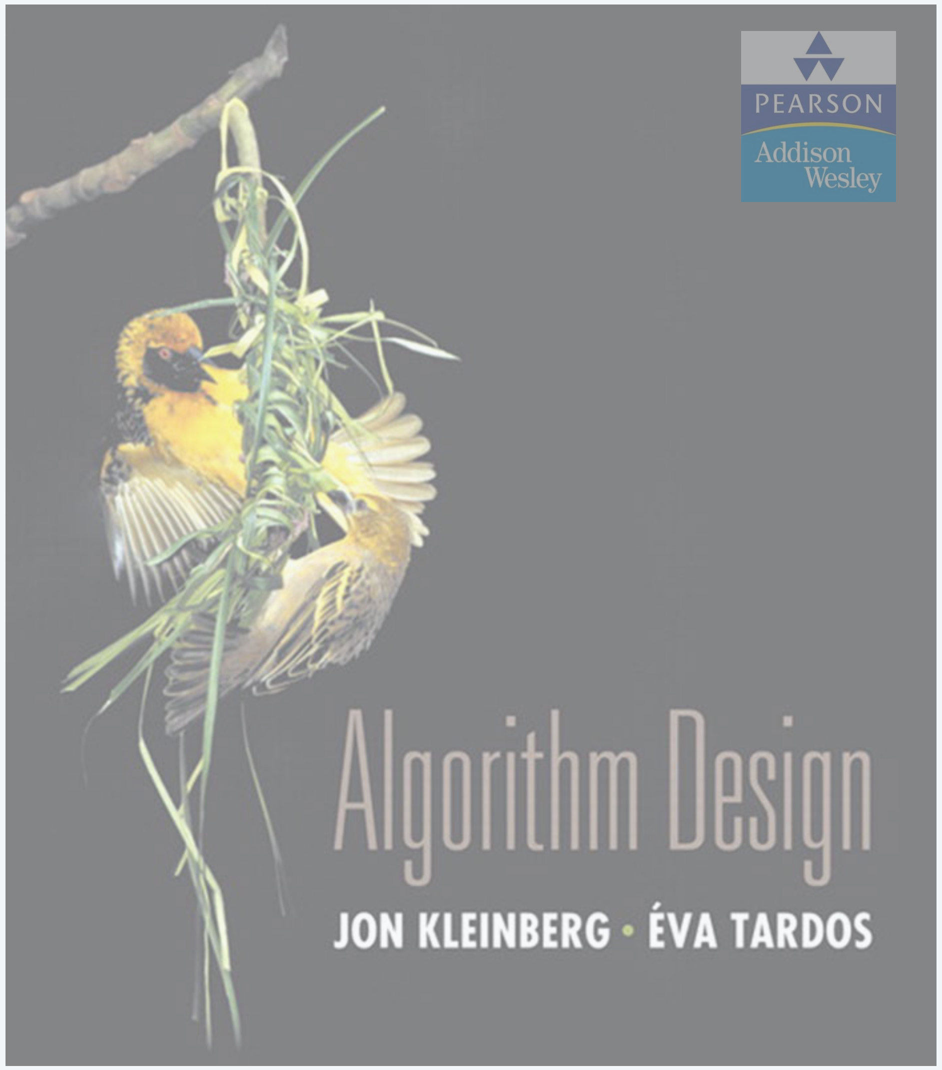
Big O notation with multiple variables

Upper bounds. $f(m, n)$ is $O(g(m, n))$ if there exist constants $c > 0$, $m_0 \geq 0$, and $n_0 \geq 0$ such that $f(m, n) \leq c \cdot g(m, n)$ for all $n \geq n_0$ and $m \geq m_0$.

Ex. $f(m, n) = 32mn^2 + 17mn + 32n^3$.

- $f(m, n)$ is both $O(mn^2 + n^3)$ and $O(mn^3)$.
- $f(m, n)$ is neither $O(n^3)$ nor $O(mn^2)$.

Typical usage. Breadth-first search takes $O(m + n)$ time to find a shortest path from s to t in a digraph with n nodes and m edges.



SECTION 2.4

2. ALGORITHM ANALYSIS

- *computational tractability*
- *asymptotic order of growth*
- *implementing Gale–Shapley*
- *survey of common running times*

Constant time

Constant time. Running time is $O(1)$.

bounded by a constant, which does not depend on input size n

Examples.

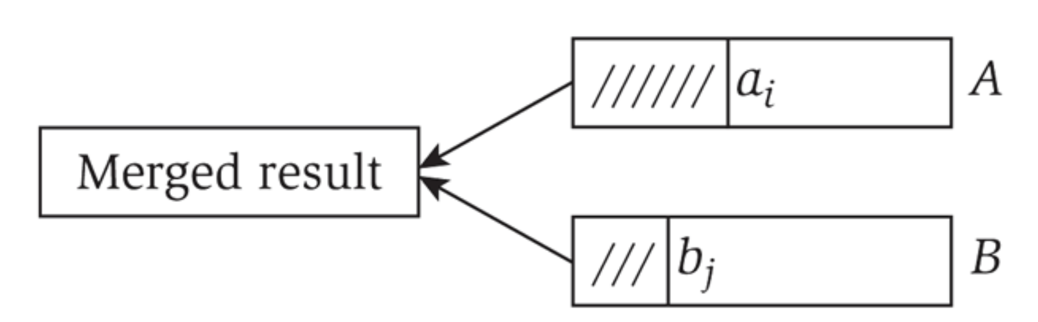
- Conditional branch.
- Arithmetic/logic operation.
- Declare/initialize a variable.
- Follow a link in a linked list.
- Access element i in an array.
- Compare/exchange two elements in an array.
- ...

Linear time

Linear time. Running time is $O(n)$.

Merge two sorted lists. Combine two sorted linked lists $A = a_1, a_2, \ldots, a_n$ and $B = b_1, b_2, \ldots, b_n$ into a sorted whole.

$O(n)$ algorithm. Merge in mergesort.



$i \leftarrow 1$; $j \leftarrow 1$.

WHILE (both lists are nonempty)

IF ($a_i \le b_j$) append a_i to output list and increment i.

ELSE append b_j to output list and increment j.

Append remaining elements from nonempty list to output list.

Logarithmic time

Logarithmic time. Running time is $O(\log n)$.

Search in a sorted array. Given a sorted array A of n distinct integers and an integer x, find index of x in array.

$O(\log n)$ algorithm. Binary search.

remaining elements

- Invariant: If x is in the array, then x is in $A[lo \mathinner{..} hi]$.
- After k iterations of WHILE loop, $(hi - lo + 1) \le n / 2^k \Rightarrow k \le 1 + \log_2 n$.

$lo \leftarrow 1$; $hi \leftarrow n$.

WHILE $(lo \le hi)$

  $mid \leftarrow \lfloor (lo + hi) / 2 \rfloor$.

  IF $(x < A[mid])$ $hi \leftarrow mid - 1$.

  ELSE IF $(x > A[mid])$ $lo \leftarrow mid + 1$.

  ELSE RETURN mid.

RETURN -1.

Linearithmic time

Linearithmic time. Running time is $O(n \log n)$.

Sorting. Given an array of n elements, rearrange them in ascending order.

$O(n \log n)$ algorithm. Mergesort.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Quadratic time

Quadratic time. Running time is $O(n^2)$.

Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest to each other.

$O(n^2)$ algorithm. Enumerate all pairs of points (with $i < j$).

```
min ← ∞.
FOR i = 1 TO n
    FOR j = i + 1 TO n
        d ← (x_i – x_j)^2 + (y_i – y_j)^2.
        IF (d < min)
            min ← d.
```

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion. [see §5.4]

Cubic time

Cubic time. Running time is $O(n^3)$.

3-SUM. Given an array of n distinct integers, find three that sum to 0.

$O(n^3)$ algorithm. Enumerate all triples (with $i < j < k$).

```
FOR i = 1 TO n
  FOR j = i + 1 TO n
    FOR k = j + 1 TO n
    IF (a_i + a_j + a_k = 0)
      RETURN (a_i, a_j, a_k).
```

Remark. $\Omega(n^3)$ seems inevitable, but $O(n^2)$ is not hard. [see next slide]

Polynomial time

Polynomial time. Running time is $O(n^k)$ for some constant $k > 0$.

Independent set of size k. Given a graph, find k nodes such that no two are joined by an edge.

k is a constant

$O(n^k)$ algorithm. Enumerate all subsets of k nodes.

```
FOREACH subset S of k nodes:
    Check whether S is an independent set.
    IF (S is an independent set)
        RETURN S.
```

- Check whether S is an independent set of size k takes $O(k^2)$ time.
- Number of k-element subsets = $$\binom{n}{k} = \frac{n(n-1)(n-2)\times\cdots\times(n-k+1)}{k(k-1)(k-2)\times\cdots\times 1} \leq \frac{n^k}{k!}$$
- $O(k^2\, n^k / k!) = O(n^k)$.

poly-time for $k = 17$,
but not practical

Exponential time

Exponential time. Running time is $O(2^{n^k})$ for some constant $k > 0$.

Independent set. Given a graph, find independent set of max cardinality.

$O(n^2 2^n)$ algorithm. Enumerate all subsets.

$S^* \leftarrow \varnothing$.

FOREACH subset S of nodes:

  Check whether S is an independent set.

  IF (S is an independent set and $|S| > |S^*|$)

    $S^* \leftarrow S$.

RETURN S^*.