Order independence of strategy elimination procedures in strategic games

Krzysztof R. Apt
Summary

• In strategic games iterated elimination of various ‘not good’ strategies has been studied.

• We provide elementary and uniform proofs of order independence for such strategy elimination procedures.

• Both pure and mixed strategies are considered.

• Crucial tools:
  – for finite games: 
    Newman’s Lemma (1942),
  – for infinite games: 
    Tarski’s Fixpoint Theorem (1955).
Strict Dominance: Intuition

Consider the following strategic game:

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<thead>
<tr>
<th></th>
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<th>M</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>2,2</td>
<td>4,1</td>
<td>1,0</td>
</tr>
<tr>
<td>C</td>
<td>1,1</td>
<td>1,3</td>
<td>1,0</td>
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<tr>
<td>B</td>
<td>0,1</td>
<td>3,4</td>
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Which strategies should the players choose?

• B is strictly dominated by T,
• R is strictly dominated by L.

By eliminating them we get:

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Now C is strictly dominated by T, so we get:

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Now M is strictly dominated by L, so we get:

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• Conclusion: the players should choose $L$ and $T$.

• Would the result be the same if initially only $R$ were eliminated?

• Could we also eliminate $C$ at the very beginning?
  ($C$ is weakly dominated by $T$.)

• Are there other meaningful ways to eliminate strategies?

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Strategic Games

A strategic game with $n$ players:

$$G := (S_1, \ldots, S_n, p_1, \ldots, p_n),$$

where

- $S_i$ is a non-empty set of strategies of player $i$,
- $p_i$ is the payoff function for player $i$, so
  $$p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}.$$

Assumptions

The players

- choose their strategies simultaneously,
- want to maximize their payoff (are rational),
- know the game and have common knowledge of each others’ rationality.
Fix a game \((S_1, \ldots, S_n, p_1, \ldots, p_n)\).

- For \(s = (s_1, \ldots, s_n)\)
  \[ s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n). \]

- \(s_i'\) is **strictly dominated** by \(s_i''\) if
  \[ \forall s_{-i} \in S_{-i} \quad p_i(s_i', s_{-i}) < p_i(s_i'', s_{-i}), \]

- \(s_i'\) is **weakly dominated** by \(s_i''\) if
  \[ \forall s_{-i} \in S_{-i} \quad p_i(s_i', s_{-i}) \leq p_i(s_i'', s_{-i}), \]
  and
  \[ \exists s_{-i} \in S_{-i} \quad p_i(s_i', s_{-i}) < p_i(s_i'', s_{-i}). \]

- \((s_1, \ldots, s_n)\) is a **Nash equilibrium** if
  \[ p_i(s_i', s_{-i}) \leq p_i(s_i, s_{-i}) \]
  for all \(i \in \{1, \ldots, n\}\) and all \(s_i'\).

**Intuition:** each \(s_i\) is a best response to \(s_{-i}\).
Elimination of Strategies

Theorem

• Elimination of strictly dominated strategies preserves all Nash equilibria.

• Elimination of weakly dominated strategies may remove some/all Nash equilibria.

• If iterated elimination of weakly dominated strategies results in one joint strategy, then this is a Nash equilibrium.
Example

Matching Pennies

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<tr>
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<th>T</th>
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<tbody>
<tr>
<td>H</td>
<td>1,−1</td>
<td>−1,1</td>
</tr>
<tr>
<td>T</td>
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• No Nash equilibrium.

Matching Pennies with an Edge

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</tr>
</tbody>
</table>

• \((E, E)\) is a unique Nash equilibrium.
• each \(E\) is weakly dominated by \(T\).
Three Examples from Economics

Common objective: profit maximization.

- Augustin Cournot (1838)
  - one product,
  - 2 companies decide simultaneously how much to produce,
  - price is decreasing in total output.

- Joseph Bertrand (1883)
  - one product,
  - 2 companies decide simultaneously the price,
  - demand is linear in the lower price.
• Harold Hotelling (1929)
  – one product,
  – 2 companies decide *simultaneously* their *location*,
  – customers choose the closest vendor.

**Example**

For instance:

Then:
\[
baker_1(3, 8) = 5,
baker_2(3, 8) = 6.
\]

Where do I put my bakery?
Then:
\[ \text{baker}_1(6, 6) = 5.5, \]
\[ \text{baker}_2(6, 6) = 5.5. \]

• (6, 6) is a unique Nash equilibrium.
• (6, 6) is the outcome of iterated elimination of strictly dominated strategies.
• Analogous conclusions hold for
  – Cournot competition,
  – Bertrand competition,
    (but these games are infinite).
Assumption

Games are Finite
Reductions of Games

• Given $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ we call $G' := (S'_1, \ldots, S'_n)$ a restriction of $G$ if $S'_i \subseteq S_i$ for $i \in [1..n]$.

• Consider a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and its restriction $G' := (S'_1, \ldots, S'_n)$.

$$G \rightarrow \_ S G'$$

when $G \neq G'$ and

$\forall i \in [1..n] \forall s''_i \in S_i \setminus S'_i \exists s'_i \in S_i$

($s''_i$ is strictly dominated by $s'_i$).

Notes

• We do not require that all strictly dominated strategies are deleted.

• Any notion of strategy dominance $D$ entails a reduction relation $\rightarrow_D$ on restrictions.

• $D$ is order independent if for each game $G$ all $\rightarrow_D$ sequences starting in $G$ have a unique outcome.
Weak Dominance

Consider

\[
\begin{array}{cc}
L & R \\
T & 1,0 & 0,0 \\
B & 0,0 & 0,1 \\
\end{array}
\]

- \(B\) is weakly dominated by \(T\),
- \(L\) is weakly dominated by \(R\).

By eliminating \(B\) we get

\[
\begin{array}{cc}
L & R \\
T & 1,0 & 0,0 \\
\end{array}
\]

But by eliminating \(L\) we get

\[
\begin{array}{c}
R \\
T & 0,0 \\
B & 0,1 \\
\end{array}
\]

**Conclusion** Weak dominance is not order independent.
Fix a game \((S_1, \ldots, S_n, p_1, \ldots, p_n)\) and strategies \(s'_i\) and \(s''_i\) of player \(i\).

- \(s'_i\) and \(s''_i\) are compatible if
  \[
  \forall j \in [1..n] \forall s_{-i} \in S_{-i} \quad p_i(s'_i, s_{-i}) = p_i(s''_i, s_{-i}) \Rightarrow p_j(s'_i, s_{-i}) = p_j(s''_i, s_{-i}).
  \]

- \(s'_i\) is nicely weakly dominated by \(s''_i\) if
  - \(s'_i\) is weakly dominated by \(s''_i\),
  - \(s'_i\) and \(s''_i\) are compatible.

- \(s'_i\) and \(s''_i\) are payoff equivalent if
  \[
  \forall j \in [1..n] \forall s_{-i} \in S_{-i} \quad p_j(s'_i, s_{-i}) = p_j(s''_i, s_{-i}).
  \]
Mixed Strategies: Intuition

Consider

\[
\begin{array}{cc}
L & R \\
T & 2, 1 & 0, 0 \\
B & 0, 0 & 1, 2 \\
\end{array}
\]

and two probability distributions, one for each player:

\[
\begin{array}{cc}
1/3 & 2/3 \\
2/3 & 2/9 & 4/9 \\
1/3 & 1/9 & 2/9 \\
\end{array}
\]

Each of them yields one mixed strategy per player:

mixed strategy of player 1: \(2/3 \cdot T + 1/3 \cdot B\),
mixed strategy of player 2: \(1/3 \cdot L + 2/3 \cdot R\).

When they are chosen

- player 1 gets
  \[
  2/9 \cdot 2 + 4/9 \cdot 0 + 1/9 \cdot 0 + 2/9 \cdot 1 = 2/3,
  \]
- player 2 gets
  \[
  2/9 \cdot 1 + 4/9 \cdot 0 + 1/9 \cdot 0 + 2/9 \cdot 2 = 2/3.
  \]
Mixed Strategies: Formally

- **Probability distribution over a finite non-empty set** $A$: a function 
  \[ \pi : A \rightarrow [0, 1] \]
  such that \( \sum_{a \in A} \pi(a) = 1 \).
- \( \Delta A \): the set of probability distributions over $A$.
- **A mixed strategy** for player $i$: probability distribution over his set of strategies.
- Consider a game \((S_1, \ldots, S_n, p_1, \ldots, p_n)\). We extend each payoff function $p_i$ to
  \[ p_i : \Delta S_1 \times \ldots \times \Delta S_n \rightarrow \mathcal{R}, \]
  by putting
  \[ p_i(m_1, \ldots, m_n) := \sum_{s \in S} m_1(s_1) \cdots m_n(s_n) \cdot p_i(s). \]
**Strict Mixed Dominance**

**Example**

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<tr>
<td><strong>B</strong></td>
<td>1, −</td>
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- **B** is neither strictly nor weakly dominated by **T** or by **M**.
- **B** is strictly dominated by the mixed strategy $1/2 \cdot T + 1/2 \cdot M$.

**Conclusion**

Mixed strategies entail new dominance notions.
Weak Confluence

- $A$ a set, $\rightarrow$ a binary relation on $A$.
  $\rightarrow^*$: the transitive reflexive closure of $\rightarrow$.
- $b$ is a $\rightarrow$-normal form of $a$ if
  - $a \rightarrow^* b$,
  - no $c$ exists such that $b \rightarrow c$.
- If each $a \in A$ has a unique normal form, then $(A, \rightarrow)$ satisfies the unique normal form property.
- $\rightarrow$ is weakly confluent if $\forall a, b, c \in A$

$$
\begin{array}{c}
a \\
\downarrow & \downarrow \\
b & c
\end{array}
\text{implies that for some } d \in A

\begin{array}{c}
b \\
\downarrow & \downarrow & \downarrow \\
* & * & d
\end{array}$$
Newman’s Lemma (’42)

Consider \((A, \rightarrow)\) such that

- no infinite \(\rightarrow\) sequences exist,
- \(\rightarrow\) is weakly confluent.

Then \(\rightarrow\) satisfies the unique normal form property.
How to Prove Unique Normal Form Property

→ is one step closed if ∀a ∈ A ∃a′ ∈ A such that a → ε a′ and ∀b ∈ A

\[
\begin{array}{c}
\text{a} \\
\searrow \downarrow \epsilon \\
\text{b} \quad \text{a}'
\end{array}
\]

implies

\[
\begin{array}{c}
\text{a} \\
\searrow \downarrow \epsilon \\
\text{b} \rightarrow \epsilon \text{a}'
\end{array}
\]

Assume: no infinite → sequences exist.

Three Ways to Prove Unique Normal Form Property:

• show that → is one step closed;
• show that → is weakly confluent;
• by finding a ‘simpler’ relation →_1 such that
  – no infinite →_1 sequences exist,
  – →_1 is weakly confluent,
  – →_1 = →+.
Summary of Results

- $S$: strict dominance,
- $W$: weak dominance,
- $NW$: nice weak dominance,
- $PE$: payoff equivalence,
- $RM$: the ‘mixed strategy’ version of the dominance relation $R$,
- $inh-R$: the ‘inherent’ version of the (mixed) dominance relation $R$.

$OI$: order independence
$\sim-OI$: order independence up to strategy renaming

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<td>Osborne and Rubinstein ’94</td>
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<td>Börgers ’90: equal to $SM$</td>
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Marx and Swinkels ’97
Part II

Infinite Games
Note (Dufwenberg and Stegeman ’02)
Strict dominance is not order independent for infinite games.

Example
Consider a two-players game $G$ with
$S_1 = S_2 = \mathcal{N}$,
$p_1(k, l) := k$,
$p_2(k, l) := l$.

Then

\[
\begin{array}{c}
G' \\
\emptyset \\
G' \\
\end{array}
\]

where $G' := (\emptyset, \emptyset)$. 

\[
G
\]
Operators

$(D, \subseteq)$: a complete lattice with the largest element $\top$.

$T$: an operator on $(D, \subseteq)$, i.e., $T : D \to D$.

- $T$ is **monotonic** if
  $$G_1 \subseteq G_2 \implies T(G_1) \subseteq T(G_2).$$

- $T$ is **contracting** if for all $G$
  $$T(G) \subseteq G.$$

- $G$ is a **fixpoint** of $T$ if $G = T(G)$.

- **Transfinite iterations** of $T$ on $D$:
  - $T^0 := \top$,
  - $T^{\alpha+1} := T(T^\alpha)$,
  - for limit ordinal $\beta$, $T^\beta := \bigcap_{\alpha < \beta} T^\alpha$,
  - $T^\infty := \bigcap_{\alpha \in \text{Ord}} T^\alpha$.

**Tarski’s Theorem** For a monotonic operator $T$ on $(D, \subseteq)$, $T^\infty$ is the largest fixpoint of $T$. 
**Order Independence**

$T$: contracting operator on complete lattice $(D, \subseteq)$. (‘$T$ removes strategies’)

- $T$ is **order independent** if
  \[ R^\infty = T^\infty \]
  (‘the outcomes of the iterated eliminations of strategies coincide’)

  for each $R$ such that for all $\alpha$
  
  - $T(R^\alpha) \subseteq R(R^\alpha) \subseteq R^\alpha$
    (‘$R$ removes from $R^\alpha$ some strategies that $T$ removes’)
  
  - if $T(R^\alpha) \subsetneq R^\alpha$, then $R(R^\alpha) \subsetneq R^\alpha$
    (‘if $T$ can remove some strategies from $R^\alpha$, then $R$ as well’).

- We call each such $R$ a **relaxation** of $T$.

- **Theorem**
  
  Every monotonic operator on $(D, \subseteq)$ is order independent.
Strict Dominance as an Operator

Fix an initial game

\[ G := (S_1, \ldots, S_n, p_1, \ldots, p_n), \]

its restriction

\[ G' := (S'_1, \ldots, S'_n), \]

and strategies \( s'_i, s''_i \in S_i \) of player \( i \).

- \( s'_i \) is strictly dominated on \( G' \) by \( s''_i \) if
  \[ \forall s_{-i} \in S'_{-i} \quad p_i(s'_i, s_{-i}) < p_i(s''_i, s_{-i}). \]

**Abbreviation:** \( s''_i \succ G' s'_i \).

- \( T_S(G') := (S''_1, \ldots, S''_n) \),
  where \( G' := (S'_1, \ldots, S'_n) \) is a restriction of \( G \) and
  \[ S''_i := \{ s_i \in S'_i \mid \neg \exists s'_i \in S'_i \ s'_i \succ G' s_i \}. \]

- \( T_S \) is not monotonic.
Limited Order Independence

Fix an initial game

\[ G := (S_1, \ldots, S_n, p_1, \ldots, p_n). \]

\( D(\alpha) \) For all relaxations \( R \) of \( T_S \)

\[
\forall i \in [1..n] \forall s_i \in S_i \\
\text{if } \exists s'_i \in S_i \ s'_i \succ_R^\alpha s_i, \\
\text{then } \exists s^*_i \in S'_i \ s^*_i \succ_R^\alpha s_i, \\
\text{where } R^\alpha := (S'_1, \ldots, S'_n).
\]

Intuition: each \( s_i \) strictly dominated on \( R^\alpha \)

is strictly dominated on \( R^\alpha \) by some strategy in \( R^\alpha \).

Theorem Assume property \( \forall \alpha D(\alpha) \).
Then \( T_S \) is order independent.

Note By Dufwenberg and Stegeman ’02 this covers the case of compact games with continuous payoffs.

Example
The mixed extension of a finite game is compact game with continuous payoffs.
Bibliography


Global Strict Dominance

Given initial game

\[ G := (S_1, \ldots, S_n, p_1, \ldots, p_n). \]

- \( T^G_{GS}(G') := (S''_1, \ldots, S''_n), \)
  where \( G' := (S'_1, \ldots, S'_n) \) is a restriction of \( G \) and
  \[ S'''_i := \{ s_i \in S'_i | \neg \exists s'_i \in S_i \ s'_i \succ_G s_i \}. \]
Global versus Local Dominance

• $T_{GS}$ is monotonic, so order independent.
• For finite games
  $$T^\infty_{GS} = T^\infty_S.$$  
• The same equalities hold for
  – strict mixed dominance,
  – weak dominance,
  – weak mixed dominance.
• For infinite games $T^\infty_{GS}$ and $T^\infty_S$ may differ,
• and likewise for the other three pairs of operators.