

**Order independence of strategy
elimination procedures in strategic
games**

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Summary

- In **strategic games** iterated elimination of various ‘not good’ strategies has been studied.
- We provide **elementary** and **uniform** proofs of order independence for such strategy elimination procedures.
- Both **pure** and **mixed** strategies are considered.
- Crucial tools:
 - for finite games:
Newman’s Lemma (1942),
 - for infinite games:
Tarski’s Fixpoint Theorem (1955).

Strict Dominance: Intuition

Consider the following strategic game:

	L	M	R
T	2, 2	4, 1	1, 0
C	1, 1	1, 3	1, 0
B	0, 1	3, 4	0, 0

Which strategies should the players choose?

- B is **strictly dominated** by T ,
- R is **strictly dominated** by L .

By eliminating them we get:

	L	M
T	2, 2	4, 1
C	1, 1	1, 3

Now C is strictly dominated by T , so we get:

	L	M
T	2, 2	4, 1

Now M is strictly dominated by L , so we get:

	L
T	2, 2

Comments

	<i>L</i>	<i>M</i>	<i>R</i>
<i>T</i>	2, 2	4, 1	1, 0
<i>C</i>	1, 1	1, 3	1, 0
<i>B</i>	0, 1	3, 4	0, 0

- Conclusion: the players should choose *L* and *T*.
- Would the result be the same if initially only *R* were eliminated?
- Could we also eliminate *C* at the very beginning?
(*C* is **weakly dominated** by *T*.)
- Are there other meaningful ways to eliminate strategies?

Strategic Games

A **strategic game** with n **players**:

$$G := (S_1, \dots, S_n, p_1, \dots, p_n),$$

where

- S_i is a non-empty, set of **strategies** of player i ,
- p_i is the **payoff** function for player i , so

$$p_i : S_1 \times \dots \times S_n \rightarrow \mathcal{R}.$$

Assumptions

The players

- choose their strategies **simultaneously**,
- want to **maximize** their payoff
(are **rational**),
- know the game and have **common knowledge** of each others' rationality.

Dominance and Nash Equilibrium

Fix a game $(S_1, \dots, S_n, p_1, \dots, p_n)$.

- For $s = (s_1, \dots, s_n)$

$$s_{-i} := (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n).$$

- s'_i is **strictly dominated** by s''_i if

$$\forall s_{-i} \in S_{-i} \ p_i(s'_i, s_{-i}) < p_i(s''_i, s_{-i}),$$

- s'_i is **weakly dominated** by s''_i if

$$\forall s_{-i} \in S_{-i} \ p_i(s'_i, s_{-i}) \leq p_i(s''_i, s_{-i}),$$

and

$$\exists s_{-i} \in S_{-i} \ p_i(s'_i, s_{-i}) < p_i(s''_i, s_{-i}).$$

- (s_1, \dots, s_n) is a **Nash equilibrium** if

$$p_i(s'_i, s_{-i}) \leq p_i(s_i, s_{-i})$$

for all $i \in \{1, \dots, n\}$ and all s'_i .

Intuition: each s_i is a best response to s_{-i} .

Elimination of Strategies

Theorem

- Elimination of strictly dominated strategies **preserves** all Nash equilibria.
- Elimination of weakly dominated strategies **may remove** some/all Nash equilibria.
- If iterated elimination of weakly dominated strategies results in one joint strategy, then this is a Nash equilibrium.

Example

Matching Pennies

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

- No **Nash equilibrium**.

Matching Pennies with an Edge

	H	T	E
H	1, -1	-1, 1	-1, -1
T	-1, 1	1, -1	-1, -1
E	-1, -1	-1, -1	-1, -1

- (E, E) is a unique **Nash equilibrium**.
- each E is **weakly dominated** by T .

Three Examples from Economics

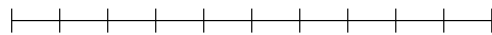
Common objective: **profit maximization**.

- Augustin Cournot (1838)
 - one product,
 - 2 companies decide **simultaneously** how much to produce,
 - price is decreasing in total output.
- Joseph Bertrand (1883)
 - one product,
 - 2 companies decide **simultaneously** the price,
 - demand is linear in the lower price.

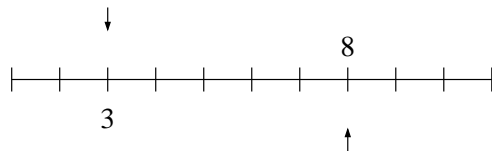
Three Examples, ctd

- Harold Hotelling (1929)
 - one product,
 - 2 companies decide **simultaneously** their **location**,
 - customers choose the closest vendor.

Example



For instance:



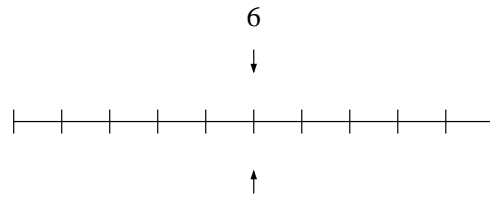
Then:

$$\text{baker}_1(3, 8) = 5,$$

$$\text{baker}_2(3, 8) = 6.$$

Where do I put my bakery?

Answer



Then:

$$\text{baker}_1(6, 6) = 5.5,$$

$$\text{baker}_2(6, 6) = 5.5.$$

- $(6, 6)$ is a unique **Nash equilibrium**.
- $(6, 6)$ is the outcome of **iterated elimination of strictly dominated strategies**.
- Analogous conclusions hold for
 - **Cournot competition**,
 - **Bertrand competition**,(but these games are infinite).

Assumption

Games are Finite

Reductions of Games

- Given $G := (S_1, \dots, S_n, p_1, \dots, p_n)$ we call

$$G' := (S'_1, \dots, S'_n)$$

a **restriction** of G if $S'_i \subseteq S_i$ for $i \in [1..n]$.

- Consider a game $G := (S_1, \dots, S_n, p_1, \dots, p_n)$ and its restriction $G' := (S'_1, \dots, S'_n)$.

$$G \rightarrow_S G'$$

when $G \neq G'$ and

$$\forall i \in [1..n] \forall s''_i \in S_i \setminus S'_i \exists s'_i \in S_i \\ (s''_i \text{ is strictly dominated by } s'_i).$$

Notes

- We do not require that **all** strictly dominated strategies are deleted.
- Any notion of strategy dominance D entails a reduction relation \rightarrow_D on restrictions.
- D is **order independent** if for each game G all \rightarrow_D sequences starting in G have a unique outcome.

Weak Dominance

Consider

	L	R
T	1, 0	0, 0
B	0, 0	0, 1

- B is **weakly dominated** by T ,
- L is **weakly dominated** by R .

By eliminating B we get

	L	R
T	1, 0	0, 0

But by eliminating L we get

	R
T	0, 0
B	0, 1

Conclusion Weak dominance is not order independent.

Other Dominance Notions

Fix a game $(S_1, \dots, S_n, p_1, \dots, p_n)$ and strategies s'_i and s''_i of player i .

- s'_i and s''_i are **compatible** if

$$\forall j \in [1..n] \forall s_{-i} \in S_{-i}$$

$$p_i(s'_i, s_{-i}) = p_i(s''_i, s_{-i}) \Rightarrow p_j(s'_i, s_{-i}) = p_j(s''_i, s_{-i})$$

- s'_i is **nicely weakly dominated** by s''_i if

- s'_i is weakly dominated by s''_i ,

- s'_i and s''_i are compatible.

- s'_i and s''_i are **payoff equivalent** if

$$\forall j \in [1..n] \forall s_{-i} \in S_{-i} p_j(s'_i, s_{-i}) = p_j(s''_i, s_{-i}).$$

Mixed Strategies: Intuition

Consider

	<i>L</i>	<i>R</i>
<i>T</i>	2, 1	0, 0
<i>B</i>	0, 0	1, 2

and two probability distributions, one for each player:

	1/3	2/3
2/3	2/9	4/9
1/3	1/9	2/9

Each of them yields **one mixed strategy** per player:

mixed strategy of player 1: $2/3 \cdot T + 1/3 \cdot B$,

mixed strategy of player 2: $1/3 \cdot L + 2/3 \cdot R$.

When they are chosen

- player 1 gets

$$2/9 \cdot 2 + 4/9 \cdot 0 + 1/9 \cdot 0 + 2/9 \cdot 1 = 2/3,$$

- player 2 gets

$$2/9 \cdot 1 + 4/9 \cdot 0 + 1/9 \cdot 0 + 2/9 \cdot 2 = 2/3.$$

Mixed Strategies: Formally

- **Probability distribution over a finite non-empty set A** : a function

$$\pi : A \rightarrow [0, 1]$$

such that $\sum_{a \in A} \pi(a) = 1$.

ΔA : the set of probability distributions over A .

- A **mixed strategy** for player i : probability distribution over his set of strategies.
- Consider a game $(S_1, \dots, S_n, p_1, \dots, p_n)$. We extend each payoff function p_i to

$$p_i : \Delta S_1 \times \dots \times \Delta S_n \rightarrow \mathcal{R},$$

by putting

$$p_i(m_1, \dots, m_n) := \sum_{s \in S} m_1(s_1) \cdot \dots \cdot m_n(s_n) \cdot p_i(s).$$

Strict Mixed Dominance

Example

	L	R
T	0, -	4, -
M	4, -	0, -
B	1, -	1, -

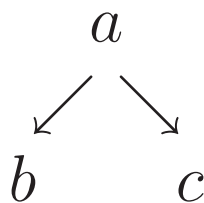
- B is neither strictly nor weakly dominated by T or by M .
- B is strictly dominated by the mixed strategy $1/2 \cdot T + 1/2 \cdot M$.

Conclusion

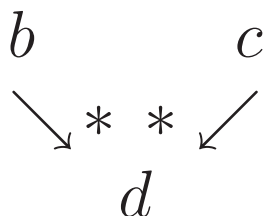
Mixed strategies entail new dominance notions.

Weak Confluence

- A a set, \rightarrow a binary relation on A .
 \rightarrow^* : the transitive reflexive closure of \rightarrow .
- b is a \rightarrow -**normal form of** a if
 - $a \rightarrow^* b$,
 - no c exists such that $b \rightarrow c$.
- If each $a \in A$ has a unique normal form, then (A, \rightarrow) satisfies the **unique normal form property**.
- \rightarrow is **weakly confluent** if $\forall a, b, c \in A$



implies that for some $d \in A$



Newman's Lemma ('42)

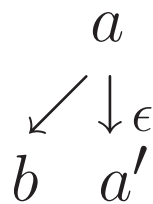
Consider (A, \rightarrow) such that

- **no infinite** \rightarrow sequences exist,
- \rightarrow is **weakly confluent**.

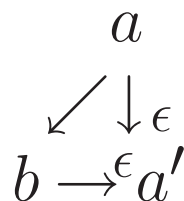
Then \rightarrow satisfies the **unique normal form property**.

How to Prove Unique Normal Form Property

\rightarrow is **one step closed** if $\forall a \in A \exists a' \in A$
such that $a \rightarrow^\epsilon a'$ and $\forall b \in A$



implies



Assume: no infinite \rightarrow sequences exist.

Three Ways to Prove Unique Normal Form Property:

- show that \rightarrow is one step closed;
- show that \rightarrow is weakly confluent;
- by finding a ‘simpler’ relation \rightarrow_1 such that
 - no infinite \rightarrow_1 sequences exist,
 - \rightarrow_1 is weakly confluent,
 - $\rightarrow_1^+ = \rightarrow^+$.

Summary of Results

- S : **strict** dominance,
- W : **weak** dominance,
- NW : **nice weak** dominance,
- PE : **payoff equivalence**,
- RM : the ‘**mixed strategy**’ version of the dominance relation R ,
- $inh-R$: the ‘**inherent**’ version of the (mixed) dominance relation R .

OI: order independence

\sim -**OI**: order independence up to strategy renaming

Dominance Notion	Property	Result originally due to
S	OI	Gilboa, Kalai and Zemel, '90 Stegeman '90
$inh - W$	OI	Börgers '90
$inh - NW$	OI	
SM	OI	Osborne and Rubinstein '94
$inh - WM$	OI	Börgers '90: equal to SM
$inh - NWM$	OI	
PE	\sim -OI	
$S \cup PE$	\sim -OI	
$NW \cup PE$	\sim -OI	Marx and Swinkels '97
PEM	\sim -OI	
$SM \cup PEM$	\sim -OI	
$NWM \cup PEM$	\sim -OI	Marx and Swinkels '97

Summary of Results

Dominance Notion	Property	Result originally due to
S	OI	Gilboa, Kalai and Zemel, '90 Stegeman '90
$inh - W$	OI	Börgers '90
$inh - NW$	OI	
SM	OI	Osborne and Rubinstein '94
$inh - WM$	OI	Börgers '90: equal to SM
$inh - NWM$	OI	
PE	\sim -OI	
$S \cup PE$	\sim -OI	
$NW \cup PE$	\sim -OI	Marx and Swinkels '97
PEM	\sim -OI	
$SM \cup PEM$	\sim -OI	
$NWM \cup PEM$	\sim -OI	Marx and Swinkels '97

Part II

Infinite Games

Strict Dominance

Note (Dufwenberg and Stegeman '02)

Strict dominance is not order independent for infinite games.

Example

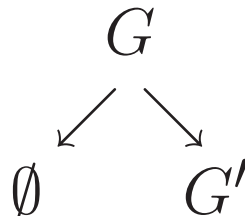
Consider a two-players game G with

$$S_1 = S_2 = \mathcal{N},$$

$$p_1(k, l) := k,$$

$$p_2(k, l) := l.$$

Then



where $G' := (\{0\}, \{0\})$.

Operators

(D, \subseteq) : a complete lattice with the largest element \top ,

T : an operator on (D, \subseteq) , i.e., $T : D \rightarrow D$.

- T is **monotonic** if

$$G_1 \subseteq G_2 \text{ implies } T(G_1) \subseteq T(G_2).$$

- T is **contracting** if for all G

$$T(G) \subseteq G.$$

- G is a **fixpoint** of T if $G = T(G)$.

- **Transfinite iterations** of T on D :

$$- T^0 := \top,$$

$$- T^{\alpha+1} := T(T^\alpha),$$

$$- \text{for limit ordinal } \beta, T^\beta := \bigcap_{\alpha < \beta} T^\alpha,$$

$$- T^\infty := \bigcap_{\alpha \in Ord} T^\alpha.$$

Tarski's Theorem For a monotonic operator T on (D, \subseteq) , T^∞ is the largest fixpoint of T .

Order Independence

T : contracting operator on complete lattice (D, \subseteq) . (' T removes strategies')

- T is **order independent** if

$$R^\infty = T^\infty$$

('the outcomes of the iterated eliminations of strategies coincide')

for each R such that for all α

$$- T(R^\alpha) \subseteq R(R^\alpha) \subseteq R^\alpha$$

(' R removes from R^α some strategies that T removes')

$$- \text{if } T(R^\alpha) \subsetneq R^\alpha, \text{ then } R(R^\alpha) \subsetneq R^\alpha$$

('if T can remove some strategies from R^α , then R as well').

- We call each such R a **relaxation** of T .

- **Theorem**

Every monotonic operator on (D, \subseteq) is order independent.

Strict Dominance as an Operator

Fix an initial game

$$G := (S_1, \dots, S_n, p_1, \dots, p_n),$$

its restriction

$$G' := (S'_1, \dots, S'_n),$$

and strategies $s'_i, s''_i \in S_i$ of player i .

- s'_i is **strictly dominated on G'** by s''_i if

$$\forall s_{-i} \in \mathbf{S}'_{-i} \quad p_i(s'_i, s_{-i}) < p_i(s''_i, s_{-i}).$$

Abbreviation: $s''_i \succ_{G'} s'_i$.

- $T_S(G') := (S''_1, \dots, S''_n)$,
where $G' := (S'_1, \dots, S'_n)$ is a restriction of G and

$$S''_i := \{s_i \in S'_i \mid \neg \exists s'_i \in S'_i \quad s'_i \succ_{G'} s_i\}.$$

- T_S is not monotonic.

Limited Order Independence

Fix an initial game

$$G := (S_1, \dots, S_n, p_1, \dots, p_n).$$

D(α) For all relaxations R of T_G

$$\forall i \in [1..n] \forall s_i \in S_i$$

$$\text{if } \exists s'_i \in S_i \ s'_i \succ_{R^\alpha} s_i,$$

$$\text{then } \exists s_i^* \in S_i \ s_i^* \succ_{R^\alpha} s_i,$$

$$\text{where } R^\alpha := (S'_1, \dots, S'_n).$$

Intuition: each s_i strictly dominated on R^α is strictly dominated on R^α by some strategy in R^α .

Theorem Assume property $\forall \alpha \mathbf{D}(\alpha)$.

Then T_G is order independent.

Note By Dufwenberg and Stegeman '02 this covers the case of compact games with continuous payoffs.

Example

The **mixed extension** of a finite game is compact game with continuous payoffs.

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- [1] K. R. Apt. Uniform proofs of order independence for various strategy elimination procedures. *The B.E. Journal of Theoretical Economics*, 4(1), 2004. (Contributions), Article 5, 48 pages.
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Global Strict Dominance

Given initial game

$$G := (S_1, \dots, S_n, p_1, \dots, p_n).$$

- $T_{GS}(G') := (S''_1, \dots, S''_n)$,
where $G' := (S'_1, \dots, S'_n)$ is a restriction of G and

$$S''_i := \{s_i \in S'_i \mid \neg \exists s'_i \in \mathbf{S}_i s'_i \succ_{G'} s_i\}.$$

Global versus Local Dominance

- T_{GS} is monotonic, so order independent.
- For finite games

$$T_{GS}^{\infty} = T_S^{\infty}.$$

- The same equalities hold for
 - strict mixed dominance,
 - weak dominance,
 - weak mixed dominance.
- For infinite games T_{GS}^{∞} and T_S^{∞} may differ,
- and likewise for the other three pairs of operators.