Talk Outline

What is game theory about?

- Game theory consists of a set of formal tools helping in understanding how decision-makers interact.
- The basic assumption here is that decision-makers act rationally and take into account their knowledge about other decision makers
- Game theory come in several guises (models) and has been exploited for the sake of analysis of rather diverse contexts

Game models

- Examples of Game models (and related applications) include:
    - Strategic games (analysis of political competitions)
    - Extensive games (analysis of bargaining and trades)
    - Repeated games (analysis of animal behavior formation through evolution)
    - Coalitional games (analysis of stable coalition formation in natural and synthetic societies)
- Each game model comes along with one or more solution concepts which have been defined in order to individuate the (form of the) outcomes of instances of that game model: for instance Nash equilibria are the reference solution concepts in strategic games

Classification of games

Groups of game models are distinguished by the nature of the interactions amongst involved players (the basic entity in all frameworks), for instance:

- Cooperative vs. Noncooperative: what are the primitive decisions? Those of individual players (Noncooperative) or those of groups of players taken as a whole (Cooperative)
- Strategic vs Extensive: what span has the decision horizon? It is just on point, so that all players' decisions are assumed to be taken simultaneously (Strategic) or it consists of a sequence of events, so that players can consider their plans of actions developed over time and contrasted with other players' previous decisions

Classification of games - continued.....

- Repeated vs. One-shot: how many times does this game will take place? Several times (So that players are possibly in the conditions to learn from past events or try to influence the other players' future behavior) (Repeated) or Only once (One-shot)
- with Perfect vs. Imperfect Information: Is each player perfectly informed about other players' moves? Yes Perfect information or No (Imperfect information)

Examples

The following table encodes the well known Prisoner's Dilemma strategic game:

	Confess	Don't Confess
Confess	-3,-3	0,-4
Don't Confess	-4,0	-1,-1

The entries represent the payoffs of the first and second player, resp. (say, the sentenced years in prison). The optimal solution would clearly be the pair of actions *(Don't Confess; Don't Confess)*, but it is not stable, since both players have an incentive to change their state. The only Nash equilibrium here is *(Confess; Confess)*

The following table encodes another well known strategic games, known as Battle of Sexes (aka, *Back or Stravinsky)*.

	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

In this case, there are two Nash equilibria, namely, *(Bach; Bach)* and *(Stravinsky; Stravinsky)*.

What is a coalitional game

- Coalitional games model situations where groups of players (Coalitions) can cooperate in order to obtain a certain worth
- Worths are assigned to coalitions and the outcome of a coalitional game is the specification of the coalition that forms and the (joint) actions it takes
- Coalition formation is determined by individual players' preference profiles over the set of possible outcomes of a game

What is a coalitional game

Therefore, a coalitional games is defined, in general, by specifying

- the set N of players in the game
- a function v, defined on the 2^N, returning the worth assigned to any coalition $s \subseteq N$
- preference relations of players in N over possible outcomes

The coalition N including all players is called the grand coalition

Applications

- Coalitional games have been extensively used to study applicative scenarios in economics and social sciences (market structure analysis, voting systems,....)
- In computer science, coalitional games are relevant, for instance, to:
  - distributed AI
  - multi-agent systems
  - electronic commerce
  - modeling and protocol design in large networks

Example

Multiple users want to route network traffic through a switch, which has a flow-dependent delay (cost). The queueing delay cost has to be shared among the users. This can be modeled as a coalitional game, where a suitable solution concept can be chosen and exploited to correspond to fair cost sharing.

Kinds of coalitional games

Coalitional games come into two main guises, depending on whether the worth of a coalition can be freely distributed amongst its members or not:

- Games with transferable payoffs (TU-Games), where the worths are transferable amongst players forming a coalition without any limitation
- Games with non-trasfereable payoffs (NTU-Games), otherwise

TU-games

Definition

A Coalitional Game with transferable payoffs is a pair $\langle N, v\rangle$ *where*

- N *is the finite set of players;*
- v *is a function that associates with every coalition* s *a real number* $v(s)$ *(the worth of* s*) (*$v : 2^N \rightarrow \Re$*).*

TU-games

- In studying coalitional TU-games, it is assumed that the grand coalition forms, for otherwise it would be meaningless to analyze fairness or stability conditions on distributions of payoffs among its members
- This assumption can be imposed by requiring the game to be cohesive

Definition

A TU-game $\langle N, v\rangle$ is cohesive if

$$v(N) \geq \sum_{k=1}^{P} v(s_k) \text{ for each partition } \{s_1, \ldots, s_P\} \text{ of } N$$

Payoff profiles

- Distributions of payoffs amongst the member of a coalition are described by vectors of reals
- For any coalition, consistent distributions are those where the sum of the distributed payoffs equals the worth assigned to that coalition

Definition

Let $n = |N|$. A profile $\bar{x}$ for N is a vector of reals $(\bar{x}_1, \ldots, \bar{x}_n)$. For a coalition $s \subseteq N$, define $\bar{x}(s) \equiv \sum_{i \in s} \bar{x}_i$. Then, $\bar{x}$ is said a s-feasible payoff profile if $\bar{x}(s) = v(s)$. Moreover, $\bar{x}$ is said a feasible payoff profile if it is an N-feasible payoff profile.

Example

An expedition of n people discover a treasure. It requires two people to carry out one piece of the treasure, in which case the value of the carried piece is equally shared between the two. For a subset s of people, the worth is given by

$$v(s) = \lfloor \frac{|s|}{2} \rfloor$$

If $|N| = 2$, $(1/2, 1/2)$ is a stable sharing. What if $|N| = 3$? Clearly $(1/2, 1/2, 0)$ is not stable, nor is $(1/3, 1/3, 1/3)$. In this case, there is no stable sharing. In general, there is no stable sharing for |N| odd, whereas the profile $(1/2, 1/2, \ldots)$ is stable for $|N|$ even.

NTU-games

Definition

A Coalitional Game without transferable payoff is a four-tuple $\langle N, X, v, (\succsim_i)_{i \in N} \rangle$*, where:*

- N *is a finite set of players;*
- X *is the set of all possible consequences;*
- $v : s \rightarrow 2^X$ *is a function that assigns, to any coalition* $s \subseteq N$ *of players, a set of consequences* $v(s) \subseteq X$*;*
- $(\succsim_i)_{i \in N}$ *is the set of all preference relations* $\succsim_i$ *on* X*,* $\forall i \in N$*.*

Solution concepts

- As in the general case, a solution concept assigns to each coalitional game a set of possible outcomes, hereby capturing a rational behavior of decision makers (the players) participating into the given game
- The stability condition, in this context, requires that the produced arrangement be immune by deviations caused by groups of players (by contrast in strategic game solutions, for instance, deviations are determined by individual players)

Solution concepts

- A relevant number of solution concepts have been defined for both TU and NTU coalitional games
- Most often, solution concept definitions are given along a set of axioms, that are proved "equivalent" with the defined solution concept

Solution concepts

- A (rather partial) list of solution concepts follows:
    - the stable set
    - the core
    - the Shapley value
    - the Banzhaf index
    - the bargaining set
    - the Kernel
    - the nucleoulus
- In the following, we are going to introduce stable sets and then focus the rest of our analysis on the core

Stable sets

- Stable sets, defined by Von Neumann and Morgenstern in 1944, are one of the oldest and best established of the solution concepts
- Each stable set Y includes distributions of the worth such that none of the members of Y is preferable to the other and each distribution not included in Y has a preferable distribution that is in Y

Stable sets

An *imputation* x is a feasible payoff profile such that $x_i >= v(\{i\})$. Let X_G be the set of imputations of the game G.

Definition

An imputation x dominates an imputation y via s, written $x \succ_s y$ if $(\forall i \in s)(x_i > y_i)$ and $x(s) \leq v(s)$. Let

$$D(Y) = \{z \in X_G \mid (\exists s)(\exists y \in Y)(y \succ_s z)\}$$

Stable sets

Definition

A stable set $Y \subseteq X_G$ is a set of imputations such that

$$Y = X_G \backslash D(Y)$$

Some properties of stable sets

- A game may have 0, 1 or more stable sets
- No stable set is a proper subset of another stable set

The core

- The core for coalitional games can be seen as an analogous of the Nash equilibrium for strategic games and it is probably the most important solution concept defined for such games
- The core forces distributions that are “stable”, i.e., no subsets of players improve their worths by leaving the grand-coalition
- Two definitions are provided next, one for TU-games, the latter for NTU-games

The core in TU-games

Definition

The core of a coalitional game with transferable payoffs $\langle N, v\rangle$ is the set of all feasible payoff profiles $\bar{x}$ such that, for all coalitions $s \subseteq N$, $\bar{x}(s) \geq v(s)$.

It follows that the core is the n-dimensional hyperspace defined by the following 2^n inequalities:

$$\begin{aligned} &\sum_{i \in s} x_i \geq v(s), \quad \forall s \subseteq N \wedge s \neq \emptyset \\ &\sum_{i \in N} x_i \leq v(N), \end{aligned}$$

where the last inequality enforces the feasibility of profiles.

The core in NTU-games

Definition

The *core of the coalitional game without transferable payoffs* $\langle N, X, v, (\succsim_i)_{i \in N} \rangle$ is the set of all $\bar{x} \in v(N)$ such that there is no coalition $s \subseteq N$ with a $\bar{y} \in v(s)$ such that $\bar{y} \succ_i \bar{x}$ for all $i \in s$.

Some properties of the core (and stable sets)

- The core may or may not exist for a given game
- Conditions have been defined for the existence of the core (cf., Bondareva-Shapley theorem)
- The core is the set of undominated imputations: $\{x \in X_G \mid (\nexists s)(\nexists y \in X_G) y \succ_s x\}$
- It then follows that:
  - The core is a subset of every stable set
  - If the core is a stable set, then it is the only stable set

Introduction

As already pointed out, the core is the most important solution concept defined for coalitional games and it is the subject of our complexity analysis

How to represent a coalitional game?

- Encoding the worth function v as a list of values is not feasible for games where the number of players is not very small
- In general, we would like to reason about complexity of the game in such a way that the input size more or less corresponds to the number of players
- It follows that a suitable non-extensive representation of the worth function v is needed

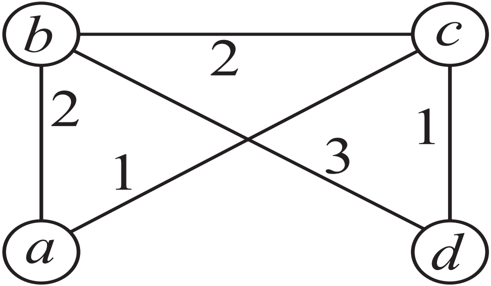
How to represent a coalitional game?

- There are a number of approaches described in the literature, including:
    - Marginal Contribution Nets: In marginal contribution nets (MC-nets), coalitional games are represented using set of rules, which are in the form *pattern* → *value*;
for instance the game of two player $N = \{a, b\}$ with worths $v(\{a\}) = 0$, $v(\{b\}) = 2$ and $v(\{a, b\}) = 7$, is represented by the following rules:

$$\{b\} \rightarrow 2 \qquad \{a \wedge b\} \rightarrow 5$$

    - Games on Graphs: In this representation, players are represented by graph vertices, and the worth of a coalition X is the sum of all arc weights in the subgraph induced by the vertices in X.

An example of graph game



The worths for all coalitions in this game are: $v(\{a\}) = 0$; $v(\{b\}) = 0$; $v(\{c\}) = 0$; $v(\{d\}) = 0$; $v(\{a, b\}) = 2$; $v(\{a, c\}) = 1$; $v(\{a, d\}) = 0$; $v(\{b, c\}) = 2$; $v(\{b, d\}) = 3$; $v(\{c, d\}) = 1$; $v(\{a, b, c\}) = 5$; $v(\{a, b, d\}) = 6$; $v(\{a, c, d\}) = 5$; $v(\{b, c, d\}) = 6$; $v(\{a, b, c, d\}) = 9$.

A general Framework for Compact Representations

We prove our results about the complexity of telling about the existence of the core for a new and general compact representation scheme of coalitional games, a scheme where it is just required the worth function to be computable in FNP, that is, computable in polynomial time by a non-deterministic Turing transducer.

A general Framework for Compact Representations

Let $\mathcal{C}$ be a class of games with transferable (resp., non-transferable) payoffs as defined by a certain given encoding scheme. Define the *worth* (*consequence*) relation for $\mathcal{C}$ as the set of tuples $W_{\mathcal{C}} = \{\langle \mathcal{G}, s, w \rangle \mid \mathcal{G} \in \mathcal{C}, v_{\mathcal{G}}(s) = w\}$ (resp., $W_{\mathcal{C}} = \{\langle \mathcal{G}, s, w \rangle \mid \mathcal{G} \in \mathcal{C}, w \in v_{\mathcal{G}}(s)\}$).

Definition

The relation $W_{\mathcal{C}}$ is polynomial-time computable if there is a positive integer k and a deterministic polynomial time transducer M that, given any game encoding $\mathcal{G} \in \mathcal{C}$ and a coalition s of players of $\mathcal{G}$, outputs a value w (resp. all consequences w) such that $\langle \mathcal{G}, s, w \rangle \in W_{\mathcal{C}}$ in at most $||\langle \mathcal{G}, s \rangle||^k$ steps.

A general Framework for Compact Representations

Definition

A worth (consequence) relation $W_{\mathcal{C}}$ *is* k*-balanced if* $||w|| \leq ||\langle \mathcal{G}, s\rangle||^k$. $W_{\mathcal{C}}$ *is said* k*-decidable if there is a non-deterministic Turing machine that decides* $W_{\mathcal{C}}$ *in at most* $||\langle \mathcal{G}, s, w\rangle||^k$ *time.*

It then follows that there is a non-deterministic Turing transducer M that may compute in $O(||\langle \mathcal{G}, s\rangle||^k)$ time the worth $v(s)$ (resp. some consequence in $v(s)$) of any coalition s of players of $\mathcal{G}$.

Definition

The relation $W_{\mathcal{C}}$ *is non-deterministically polynomial-time computable if there is a positive integer* k *such that* $W_{\mathcal{C}}$ *is* k*-balanced and* k*-decidable.*

A general Framework for Compact Representations

Definition

Let $\mathcal{C}(\mathcal{R})$ be the class of all games encoded according to some compact representation $\mathcal{R}$. Then, $\mathcal{R}$ is called

FP-representation , *if the worth relation for $\mathcal{C}(\mathcal{R})$ is polynomial-time computable.*

FNP-representation , *if the worth relation for $\mathcal{C}(\mathcal{R})$ is non-deterministically polynomial-time computable.*

Complexity of the core

- For all the above mentioned compact representations, checking whether the core is not empty is co-NP-hard.
- However, membership is not easily established, for this problem and, in fact, it was left as an open problem by several authors to settle the precise complexity of core non-emptiness for reasonable and general compact representation schemes (e.g., for MC-nets).

Complexity of the core

- We provide an answer to this question (in both the transferable and the non-transferable payoffs cases) in the rather general setting of those compact game representations satisfying the (quite weak) constraint that the associated worth function is computable in FNP and, thus, as a special case, in polynomial time (as for games on graphs and marginal contribution nets illustrated before).

Main Results: The Complexity of Core non-emptiness

In fact, we were able to show the following:

	FP representation (e.g. MC Nets, Games on Graphs)	FNP representation (e.g. NTU MC Nets)
TU Games	co-NP-complete	co-NP-complete
NTU Games	co-NP-complete	Σ_2^P-complete

The main result

Given that the co-NP-hardness of the problem at hand immediately follows from previous known results, our main result is summarized in the following theorem:

Theorem

Let $\mathcal{R}$ be a non-deterministic polynomial-time compact representation. Given any coalitional game with transferable payoffs $\mathcal{G} \in \mathcal{C}(\mathcal{R})$, deciding whether the core of $\mathcal{G}$ is not empty is in $\mathrm{co\text{-}NP}$. The same holds for polynomial-time compact representation.

A corollary

The above result also settles the precise complexity of the core non-emptiness problem for marginal contribution nets (left open by Ieong and Shoham in their EC’05 paper).

Corollary

Given a coalitional game with transferable payoffs encoded as a marginal contribution net, deciding whether its core is not empty is co-NP*-complete.*

Proof sketch

Recall that the core is the n-dimensional hyperspace defined by the following 2^n inequalities:

$$\sum_{i \in s} x_i \geq v(s), \quad \forall s \subseteq N \wedge s \neq \emptyset \tag{2.1}$$

$$\sum_{i \in N} x_i \leq v(N), \tag{2.2}$$

where the last inequality enforces the feasibility of profiles.

Proof sketch

- Because of the previous system of inequalities, the core of a TU game with n players is a polyhedral set of $\Re^n$.
- Our proof relies on showing that, for any game with transferable payoffs with an empty core, there is a small infeasibility certificate for it.

Proof sketch

It can be noted that coalitions correspond to the inequalities (2.1) and hence with the associated half-spaces of $\Re^n$. The intersection of the half-spaces associated with a set of coalitions (inequalities) S is denoted $\mathrm{Pol}(S)$.

Definition

Let $\mathcal{G} = \langle N, v \rangle$ be a game with transferable payoffs. A coalition set $S \subseteq 2^N$ is a *certificate of emptiness* (or *infeasibility certificate*) for the core of $\mathcal{G}$ if the intersection of $\mathrm{Pol}(S)$ with the grand-coalition halfspace (2.2) is empty.

Proof sketch

Previous definition is motivated by the following observation. Let P be the polyhedron of $\Re^n$ obtained as the intersection of all halfspaces (2.1). Since S is a subset of all possible coalitions, $P \subseteq \mathrm{Pol}(S)$. Therefore, if the intersection of $\mathrm{Pol}(S)$ with the grand-coalition halfspace (2.2) is empty, the intersection of this halfspace with P is empty, as well.

Proof sketch

Our main theorem then follows from the following result, which is also interesting on its own:

Theorem

Let $\mathcal{G} = \langle N, v \rangle$ be a game with transferable payoffs. If the core of $\mathcal{G}$ is empty, there is a certificate of emptiness S for it such that $|S| \leq |N|$.

The proof of the above theorem on infeasibility certificates can be also obtained as a consequence of Helly's Theorem on the intersection of families of convex sets, whose proof relies on algebraic techniques. The proof reported in our IJCAI'07 paper is a constructive one, though, based on polyhedral geometry.

Proof sketch

Therefore, for a TU-game $\mathcal{G} = \langle N, v \rangle$ with a polynomial-time deterministic representation, a non-deterministic Turing machine may check in polynomial time that the core is empty by:

- guessing the set S
- computing (in deterministic polynomial time) the worth $v(s)$, for each $s \in S$, and for the grand-coalition N
- checking that $\mathrm{Pol}(S) \cap H_P^+ = \emptyset$, where H_P^+ is the halfspace defined by the grand-coalition inequality (2.2), which is tantamount to solving a linear system consisting of $n + 1$ inequalities.

For non-deterministic polynomial-time compact representations the proof is a simple variation of the previous one.

The NTU-game case

For games where the payoffs cannot be transferred among the players, core non-emptiness turns out to be harder than in the case of transferable payoffs we have studied above.

The complexity of the core for NTU-games

Theorem

Let $\mathcal{R}$ be a non-deterministic polynomial-time compact representation. Given any coalitional game with non-transferable payoffs $\mathcal{G} \in \mathcal{C}(\mathcal{R})$. Then deciding whether the core of $\mathcal{G}$ is not empty is:

- Σ_2^P*-complete, if $\mathcal{R}$ is non-deterministic polynomial-time compact representation*
- *in* $\mathrm{co\text{-}NP}$*, if $\mathcal{R}$ is a deterministic polynomial-time compact representation*

The Complexity of Core non-emptiness

Again, a summary of the complexity results:

	FP representation (e.g. MC Nets, Games on Graphs)	FNP representation (e.g. NTU MC Nets)
TU Games	co-NP-complete	co-NP-complete
NTU Games	co-NP-complete	Σ_2^P-complete

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