# Proofs

- A proof is a mechanically derivable demonstration that a formula logically follows from a knowledge base.
- Given a proof procedure, $KB \vdash g$ means $g$ can be derived from knowledge base $KB$.
- Recall $KB \models g$ means $g$ is true in all models of $KB$.
- A proof procedure is sound if $KB \vdash g$ implies $KB \models g$.
- A proof procedure is complete if $KB \models g$ implies $KB \vdash g$.

# Bottom-up Ground Proof Procedure

One rule of derivation, a generalized form of *modus ponens*:

> *If "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" is a clause in the knowledge base, and each $b_i$ has been derived, then $h$ can be derived.*

This is forward chaining on this clause.
(This rule also covers the case when $m = 0$.)

# Bottom-up proof procedure

$KB \vdash g$ if $g \in C$ at the end of this procedure:

$C := \{\}$;
**repeat**
  **select** clause "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" in $KB$ such that
    $b_i \in C$ for all $i$, and
    $h \notin C$;
  $C := C \cup \{h\}$
**until** no more clauses can be selected.

# Example

$$a \leftarrow b \wedge c.$$
$$a \leftarrow e \wedge f.$$
$$b \leftarrow f \wedge k.$$
$$c \leftarrow e.$$
$$d \leftarrow k.$$
$$e.$$
$$f \leftarrow j \wedge e.$$
$$f \leftarrow c.$$
$$j \leftarrow c.$$

# Soundness of bottom-up proof procedure

If $KB \vdash g$ then $KB \models g$.

- Suppose there is a $g$ such that $KB \vdash g$ and $KB \not\models g$.
- Then there must be a first atom added to $C$ that isn't true in every model of $KB$. Call it $h$. Suppose $h$ isn't true in model $I$ of $KB$.
- There must be a clause in $KB$ of form

$$h \leftarrow b_1 \wedge \ldots \wedge b_m$$

  Each $b_i$ is true in $I$. $h$ is false in $I$. So this clause is false in $I$. Therefore $I$ isn't a model of $KB$.
- Contradiction.

# Fixed Point

- The $C$ generated at the end of the bottom-up algorithm is called a fixed point.
- Let $I$ be the interpretation in which every element of the fixed point is true and every other atom is false.
- $I$ is a model of $KB$.
  Proof: suppose $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in $KB$ is false in $I$. Then $h$ is false and each $b_i$ is true in $I$. Thus $h$ can be added to $C$. Contradiction to $C$ being the fixed point.
- $I$ is called a Minimal Model.

# Completeness

**If $KB \models g$ then $KB \vdash g$.**

- Suppose $KB \models g$. Then $g$ is true in all models of $KB$.
- Thus $g$ is true in the minimal model.
- Thus $g$ is in the fixed point.
- Thus $g$ is generated by the bottom up algorithm.
- Thus $KB \vdash g$.