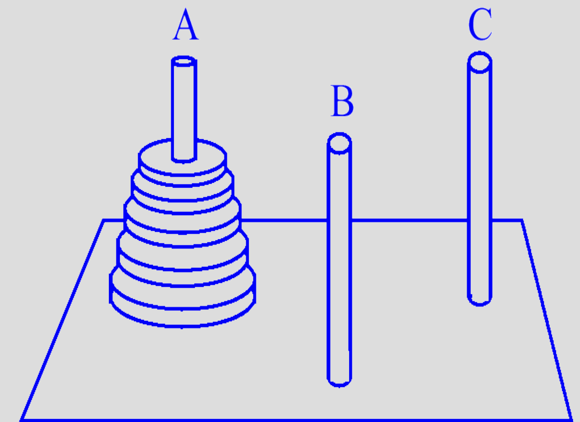
Data Structures and Algorithms
Week 11

1. Tractable and intractable problems.
2. What is a ”reasonable” running time?
3. NP problems, examples
4. NP-complete problems
5. Polynomial reducability

Towers of Hanoi/1

- **Goal:** transfer all n disks from peg A to peg C
- Rules:
  - move one disk at a time
  - never place larger disk above smaller one
- Recursive solution:
  - transfer $n - 1$ disks from A to B
  - move largest disk from A to C
  - transfer $n - 1$ disks from B to C
- Total number of moves:
  - $T(n) = 2T(n - 1) + 1$



Towers of Hanoi/2

- Recurrence:

  $T(n) = 2\,T(n-1) + 1$

  $T(1) = 1$

- Solution by repeated substitution:

  $T(n) = 2\,(2\,T(n-2) + 1) + 1 =$
  $= 4\,T(n-2) + 2 + 1 =$
  $= 4\,(2\,T(n-3) + 1) + 2 + 1 =$
  $= 8\,T(n-3) + 4 + 2 + 1 = \ldots$
  $= 2^i\,T(n-i) + 2^{i-1} + 2^{i-2} + \ldots + 2^1 + 2^0$

- The expansion stops ($n - i = 1$) when $i = n - 1$

  $T(n) = 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^1 + 2^0$

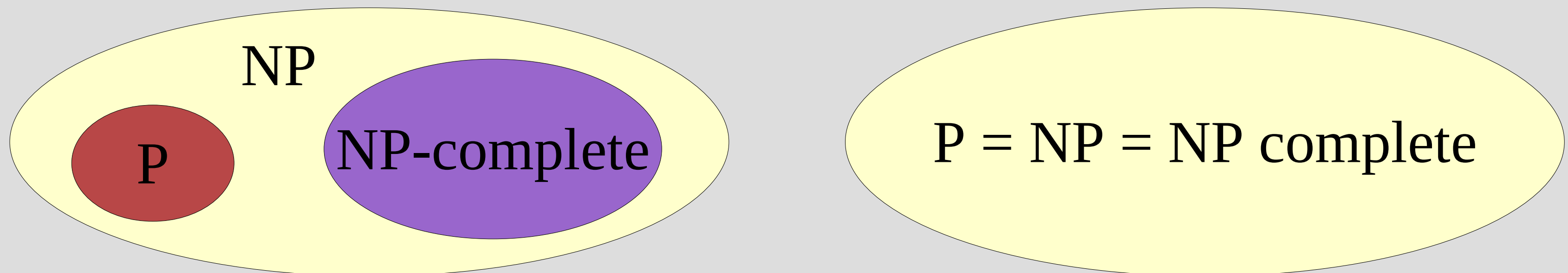
Towers of Hanoi/3

- This is a **geometric series** so that we have

  $T(n) = 2^n - 1 = O(2^n)$
- The running time of this algorithm is **exponential** (k^n).
- Good or bad news?
  - the Tibetan priests were confronted with a tower problem of 64 rings.
  - Assuming the priests move **one ring per second**, it would take ~**585 billion years** to complete the process.

Overview

- Problem class overview:
  - P: Problems solvable in polynomial time
  - NP: Problems verifiable in polynomial time
  - NPC: ”reference problems” in NP (can be used to solve the other NP problems)
- Most important open problem in theoretical computer science.
- Clay institute of mathematics offered 1 million dollar millenium prize for solution!



Decision Problems

- Are long running times linked to the size of the solution of an algorithm?
  - No! To show this we consider only TRUE/FALSE or yes/no problems in the following – **decision problems**
- We can usually transform an *optimization problem* into an **easier** *decision problem*:
  - *Optimization* problem O : "Find a shortest path between vertices u and v in a graph G."
  - Related *decision* problem D : "Determine if there is a path between vertices u and v in a graph G shorter than k."
  - If we have an easy way to solve O, we can use it to solve D.
  - If we show that D is hard, we know that O must be hard as well.

LCS

- Longest common subsequence
  - x = "sariempiolcewe"
  - y = "westigmupsalrte"
- Solution: Proceed from the end of the strings and
  - If $x_m = y_n$ append symbol to LCS(x_{m-1}, y_{n-1})
  - If $x_m \neq y_n$
    - Skip last symbol from *x or*
    - last symbol from *y*
    - Decide which symbol to skip by comparing LCS(x_m, y_{n-1}) and LCS(x_{m-1}, y_n)

LCS Implementation/1

```
int lcsRec(int i, int j) {
  if (i==0 || j==0) return 0;
  else if (x[i] == y[j]) return lcsRec(i-1,j-1) + 1;
  else return max(lcsRec(i-1,j),lcsRec(i,j-1));
}
```

Recurrence for time complexity:

$T(2n) = T(2n-2)+1$ if $x[n] = y[n]$

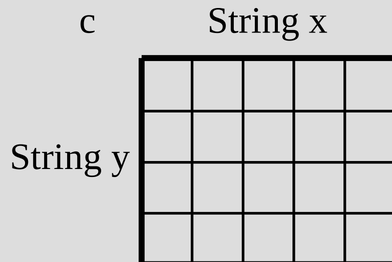
$T(2n) = 2T(2n-1)$ if $x[n] \mathrel{!}= y[n]$

$T(2n) = 1$ if $n = 0$

$$
\begin{aligned}
T(2n) &= 2T(2n-1) \\
&= 2(2T(2n-2)) \\
&= 4T(2n-2) \\
&= 4(2T(2n-3)) \\
&= 8T(2n-3) \\
&= 2^i T(2n-i) \\
&= 2^{2n} = 4^n
\end{aligned}
$$

LCS Implementations/2

```
int lcsMemo(int i, int j) {
  if (c[i][j] != -1) return c[i][j];
  else if (x[i] == y[j]) {
    c[i][j] = lcsMemo(i-1,j-1) + 1;
    return c[i][j];
  }
  else {
    c[i][j] = max(lcsMemo(i-1,j),lcsMemo(i,j-1));
    return c[i][j]; }
  }
```



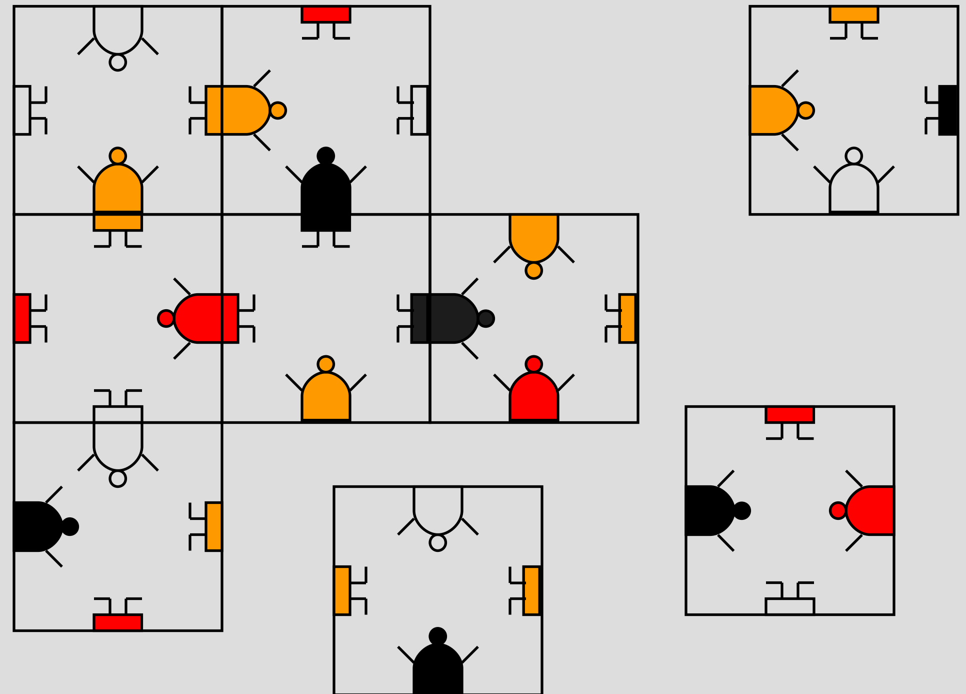
$T(n) = O(n^2)$

Runtime Comparison

N	Recursive	Memo
10	0.004	0.002
15	0.774	0.002
17	4.760	0.003
18	11.216	0.003
20	154.464	0.003
100		0.005

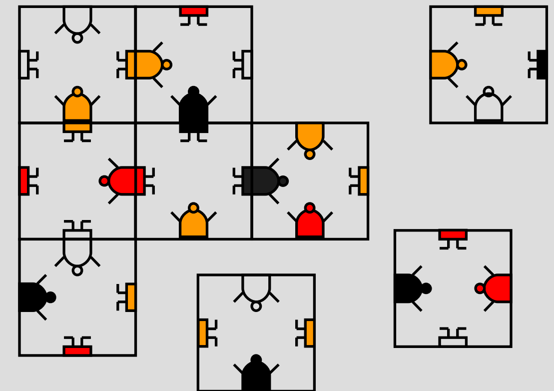
Monkey Puzzle/1

- Monkey puzzle: an example of a decision problem
- Nine square cards with imprinted "monkey halves"
- The goal is to arrange the cards in 3x3 square with matching halves.



Monkey Puzzle/2

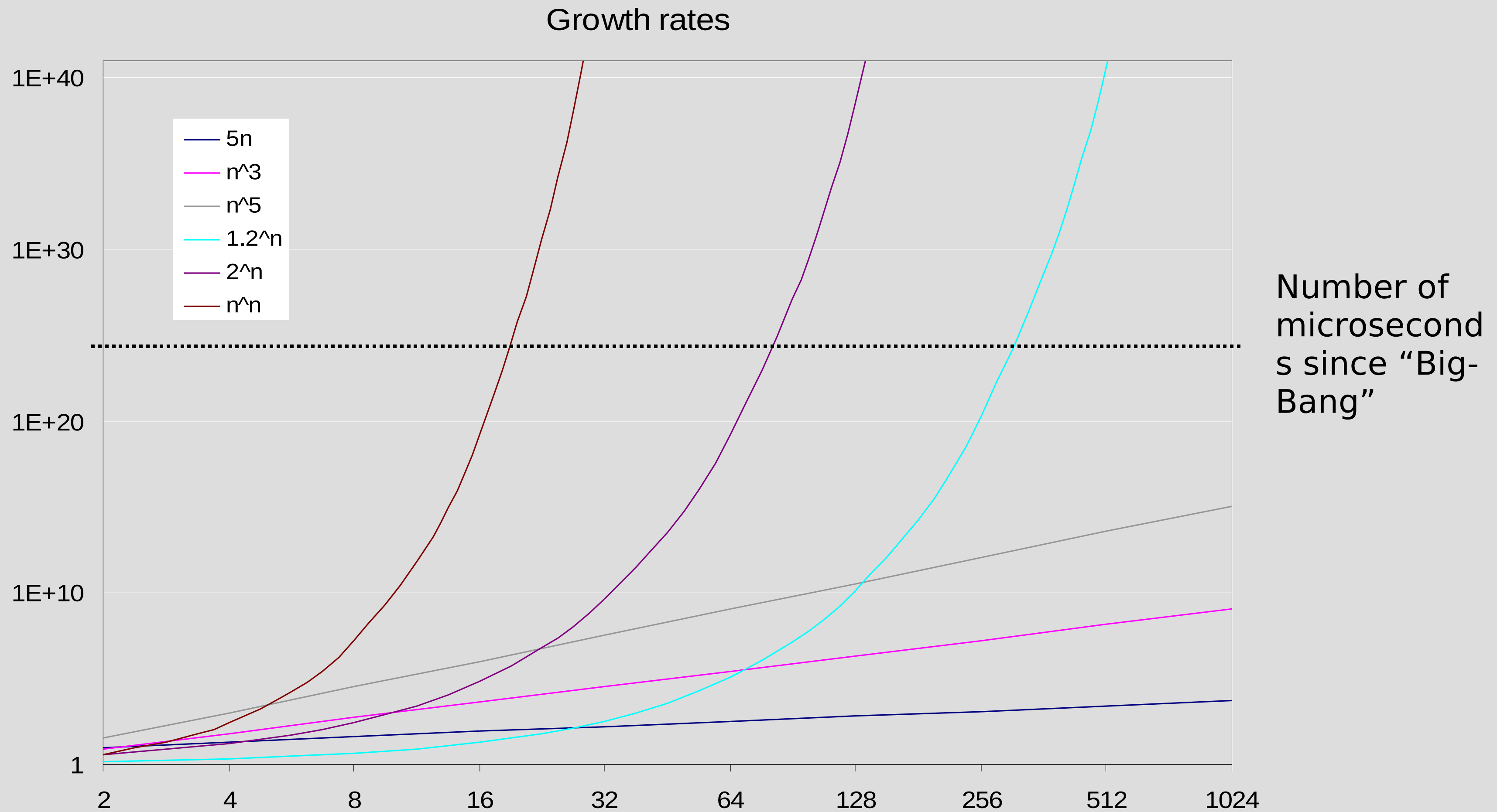
- Assumption: orientation is fixed
- *Does any MxM arrangement exist that fulfills the matching criterion?*
- Brute-force algorithm would take ***n*! time** to verify whether a solution exists
  - assuming $n = 25$, it would take 490 billion years on a one-million-per- second arrangements computer to verify whether a solution exists



Monkey Puzzle/3

- Improving the algorithm
  - discarding partial arrangements
  - etc.
- A smart algorithm would still take a couple of thousand years in the worst case
- Is there an easier way to find solutions? MAYBE! But nobody has found them, yet!

Reasonable vs. Unreasonable/1

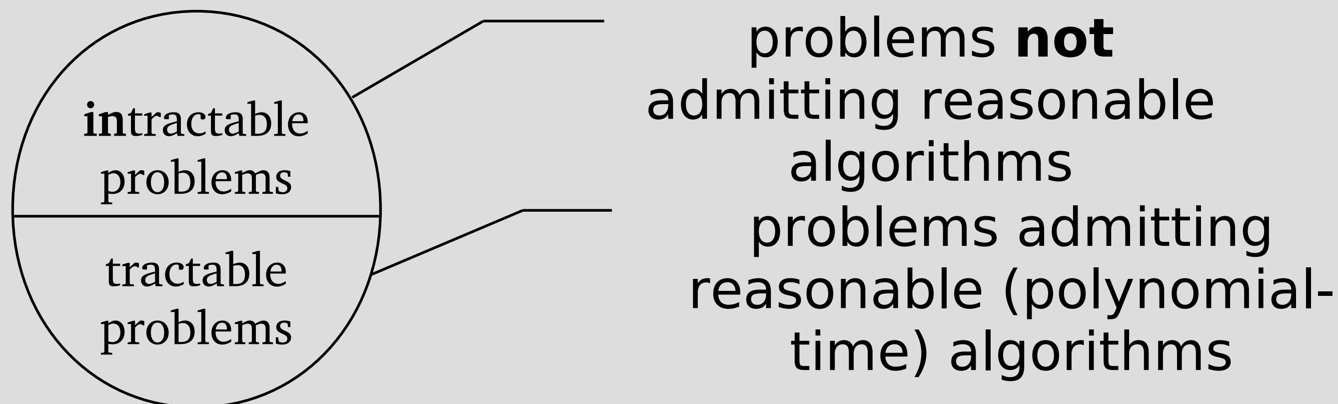


Reasonable vs. Unreasonable/2

	n/ function	10	20	50	100	300
Polynomial	n^2	1/10'000 second	1/2'500 second	1/400 second	1/100 second	9/100 second
	n^5	1/10 second	3.2 seconds	5.2 minutes	2.8 hours	28.1 days
Exponential	2^n	1/1000 second	1 second	35.7 years	400 trillion centuries	a 75 digit-number of centuries
	n^n	2.8 hours	3.3 trillion years	a 70 digit-number of centuries	a 185 digit-number of centuries	a 728 digit-number of centuries

Reasonable vs. Unreasonable/3

- "Good", reasonable algorithms
  - algorithms bound by a polynomial function n^k
  - **Tractable problems**
- "Bad", unreasonable algorithms
  - algorithms whose running time is above n^k
  - **Intractable problems**

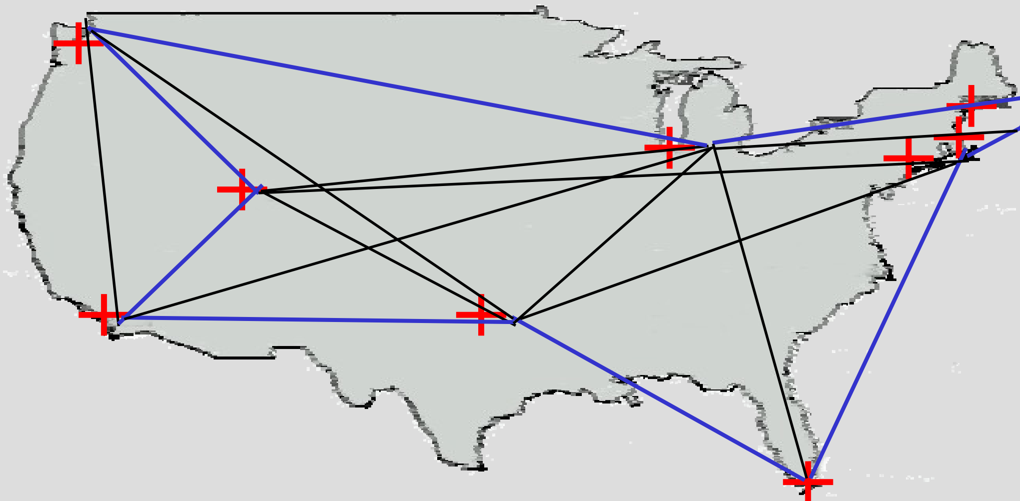


So What!

- Computers become faster every day
  - insignificant (a constant) compared to exponential running time
- Maybe the Monkey puzzle is just one specific problem, we could simply ignore
  - the monkey puzzle falls into a category of problems called NPC (NP complete) problems (~1000 problems)
  - all admit **unreasonable** solutions
  - **not known** to admit **reasonable** ones

Traveling Salesman Problem

- A traveling salesperson needs to visit *n* cities
- Is there a route of at most *d* length? (decision problem)
  - Optimization-version asks to find a shortest cycle visiting all vertices once in a weighted graph

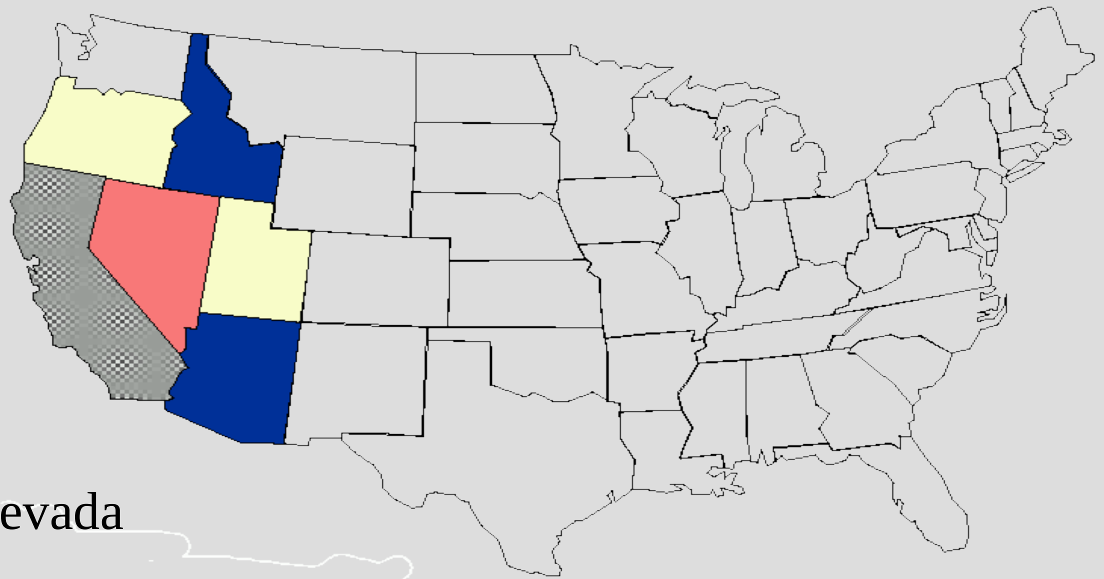


TSP Algorithms

- Naive solutions take $n!$ time in worst-case, where n is the number of edges of the graph
- No polynomial-time algorithms are known
  - TSP is an NP-complete problem
  - Simpler Hamiltonian Cycle problem is NP-complete
- Longest Path problem between A and B in a weighted graph is also NP-complete
  - Remember the running time for the shortest path problem

Coloring Problem/1

- **3-color**: given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?



NO instance
Impossible to 3-color Nevada and bordering states!

Coloring Problem/2

- Any map can be **4-colored**
- Maps that contain no points that are the junctions of an odd number of states can be **2-colored**
- No polynomial algorithms are known to determine whether a map can be **3-colored**: it's an NP-complete problem

Determining Truth (SAT)/1

- Determine the truth or falsity of logical sentences in **propositional calculus**
- Using the logical operators ($\wedge$-and, $\vee$-or, $\neg$-not) we compose expressions such as the following

  $(\neg F \vee E) \wedge (F \vee D) \wedge (\neg E \vee F \vee \neg D)$
- The algorithmic problem calls for determining the **satisfiability** of such sentences
  - e.g., D = true, E and F = false

Determining Truth (SAT)/2

- Exponential time algorithm on n = number of distinct variables ($O(2^n)$)
  - Best known solution for the general SAT problem
  - SAT problem is in the NP-complete class.
- Good performance if we only have Horn clauses (at most 1 positive literal)

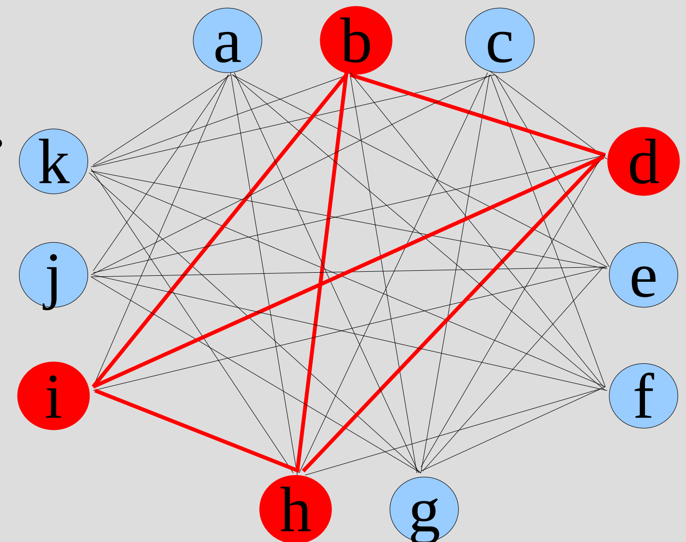
$$(\neg x \vee \neg y \vee \neg z) \wedge (\neg x \vee x)$$

- Good performance if if all clauses have only two literals (2SAT)

$$(\neg F \vee E) \wedge (F \vee D) \wedge (\neg E \vee F)$$

CLIQUE

- Given *n* people and their pairwise relationships, is there a group of *s* people (called *clique*) such that every pair in the group knows each other
  - people: a, b, c, …, k
  - friendships: (a,e), (a,f),…
  - clique size: s = 4?
  - YES, {b, d, i, h} is a *certificate*!



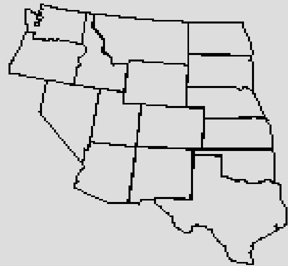
P

- Definition of P:
  - Set of all decision problems solvable in polynomial time on *real* computers
- Examples:
  - SHORTEST PATH: Is the shortest path between u and v in a graph shorter than k?
  - RELPRIME: Are the integers x and y relatively prime?
    - YES: $(x, y) = (34, 39)$.
  - LCS: Given two strings x and y, is the length of their longest common subsequence $> k$?
    - YES: (x, y, k) = ("CGTTAG", "GGTACG", 3)

Certificates/1

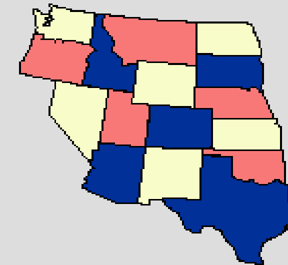
- To find a solution for an NPC problem, we seem to be required to try out exponential amounts of partial solutions.
- Failing in extending a partial solution requires backtracking.
- However, once we found a solution, convincing someone of it is easy, if we keep a **certificate** (i.e., truth values for SAT, vertices for CLIQUE).
- The problem is finding an answer (exponential), but not verifying a potential solution (polynomial).

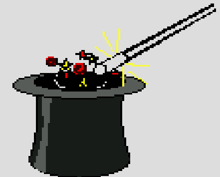
Certificates/2

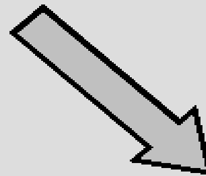


Input x:

Certificate c:

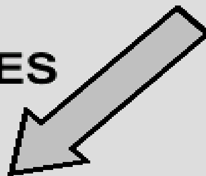




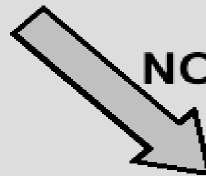


Verifier:
1. Check that x and c describe same map.
2. Count number of distinct colors in c.
3. Check all pairs of adjacent states.

YES



NO



x is a YES instance

no conclusion

3-COLOR is in NP.

Magic Coins and Oracles

- Assume we use a magic coin in the backtracking algorithm
  - whenever it is possible to extend a partial solutions in ”two” ways, we toss a magic coin (two monkey cards, next truth assignment, etc.)
  - the outcome of this ”act” determines further actions: we use magical insight.
- Such algorithms are termed ”**non-deterministic**”
  - they **guess** which option is better, **rather** than employing some **deterministic procedure** to go through the alternatives

NP/1

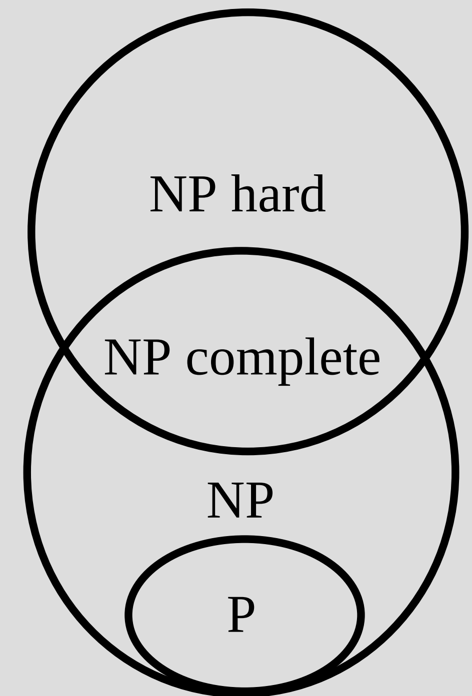
- Definition of NP:
  - Set of all decision problems solvable in polynomial time on a *non-deterministic* computer
- Definition important because it links many fundamental problems
- Useful alternative definition
  - Set of all decision problems with efficient verification algorithms
    - efficient = polynomial number of steps on deterministic TM
  - Verifier: algorithm for decision problem with extra input

NP/2

- NP = set of decision problems with efficient verification algorithms
- Why doesn’t this imply that all problems in NP can be solved efficiently?
  - PROBLEM: need to know certificate ahead of time
    - real computers can simulate by guessing all possible certificates and verifying
    - naïve simulation takes exponential time unless you get "lucky"

NP-Completeness/1

- **NP-complete** problems are the toughest (hardest) problems in NP.
- **NP-hard** problems
  - are at least as hard as the hardest problems in NP and
  - an NP-complete problem can be reduced in polynomial time.



NP-Completeness/2

- Each NPC problem's faith is tightly coupled to all the others (complete set of problems).
- Finding a **polynomial time algorithm for one NPC problem** would **automatically** yield a polynomial time algorithm **for all NP problems**
- Proving that one NP-complete problem has an **exponential lower bound** would **automatically** prove that **all other NP-complete** problems have exponential lower bounds.

Polynomial Time Reduction

- *How can we prove such a statement?*
- **Polynomial time reduction:**
  - given two problems
  - it is an algorithm running in polynomial time that reduces one problem to the other such that
    - given input X to the first and asking for a yes/no answer
    - we transform X into input Y to the second problem such that its answer matches the answer of the first problem

Reduction Example/1

- Reduction is a general technique for showing that one problem is harder (easier) than another
  - For problems A and B, we can often show: if A can be solved efficiently, then so can B
  - In this case, we say B reduces to A (B is ”easier” than A, or, B cannot be ”worse” than A)
- NP-complete problem A:
  - It is NP problem
  - Any NP problem B can be *polynomialy* reduced to A

Redcution Example/2

- SAT reduces to CLIQUE
  - Given any input to SAT, we create a corresponding input to CLIQUE that will help us solve the original SAT problem
  - Specifically, for a SAT formula with K clauses, we construct a CLIQUE input that has a clique of size K if and only if the original Boolean formula is satisfiable
  - If we had an efficient algorithm for CLIQUE, we could apply our transformation, solve the associated CLIQUE problem, and obtain the yes/no answer for the original SAT problem

Reduction Example/3

- SAT reduces to CLIQUE
  - Associate a person to each variable occurrence in each clause

x' y z

x' x

y' y'

z' z

y z

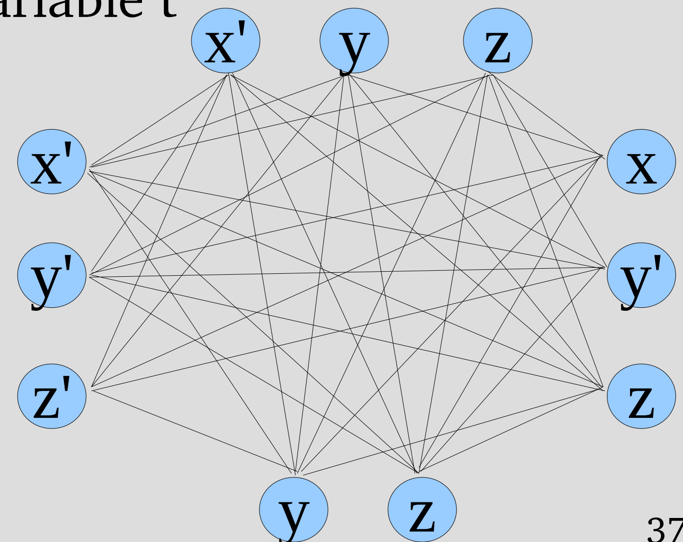
Boolean formula:
(x' + y + z)(x + y' + z)(y + z)(x'+y'+z')

Reduction Example/4

- SAT reduces to CLIQUE
  - Associate a person to each variable occurrence in each clause
  - ”Two people” know each other except if:
    - they come from the same clause
    - they represent t and t’ for some variable t

Boolean formula:

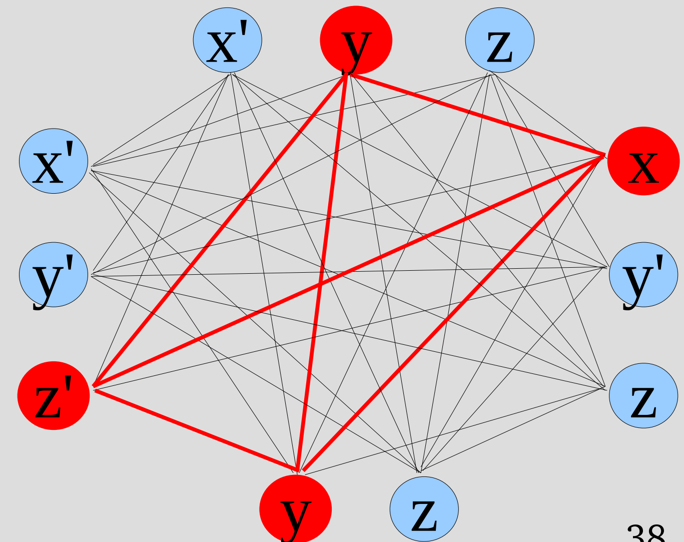
(x' + y + z)(x + y' + z)(y + z)(x'+y'+z')



Reduction Example/5

- SAT reduces to CLIQUE
  - Two people know each other except if:
    - they come from the same clause
    - they represent t and t' for some variable t
  - Clique of size 4 $\Rightarrow$ satisfiable assignment
    - set variable in clique to "true"
    - (x, y, z) = (true, true, false)

Boolean formula:
(x' + y + z)(x + y' + z)(y + z)(x'+y'+z')

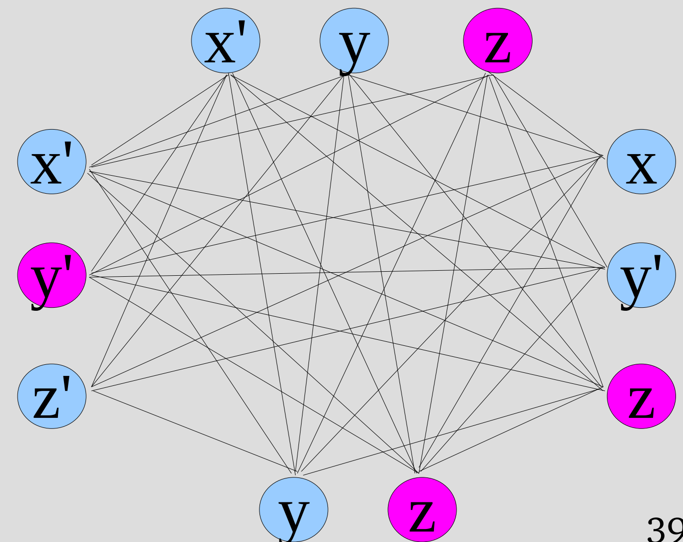


Reduction Example/6

- SAT reduces to CLIQUE
  - Two people know each other except if:
    - they come from the same clause
    - they represent t and t’ for some variable t
  - Clique of size 4 ⇒ satisfiable assignment
  - Satisfiable assignment ⇒ clique of size 4
    - (x, y, z) = (false, false, true)
    - choose one true literal from each clause

Boolean formula:

(x' + y + z)(x + y' + z)(y + z)(x'+y'+z')



CLIQUE is NP-complete

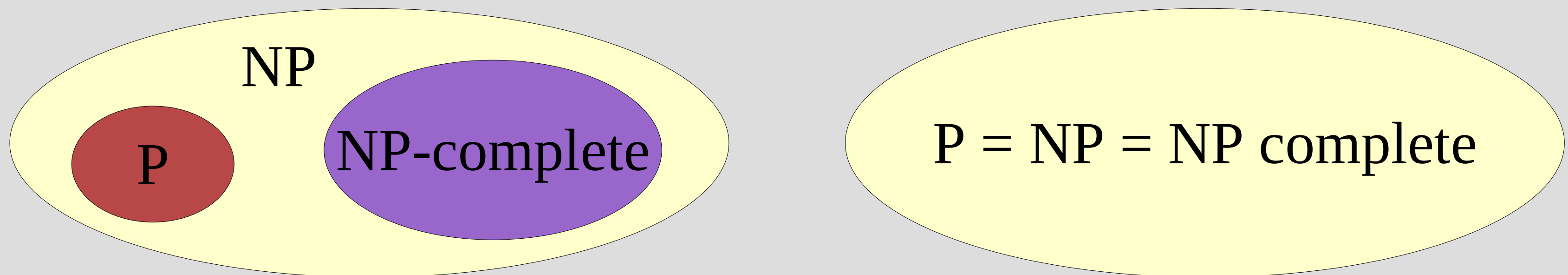
- CLIQUE is NP-complete
  - CLIQUE is in NP
  - SAT is in NP-complete
  - SAT reduces to CLIQUE
- Hundreds of problems can be shown to be NP-complete that way

The Start

- The world's first NP-complete problem
- SAT is NP-complete (Cook-Levin, 1960's).

The Main Question/1

- Is P = NP?
  - Is the original DECISION problem as easy as VERIFICATION?
- Most important open problem in theoretical computer science. Millenium prize from Clay institute.



The Main Question/2

- If P=NP, then:
  - Efficient algorithms for 3-COLOR, TSP, and factoring.
  - Modern cryptography breaks down on conventional machines
- If no then:
  - Can't hope to write efficient algorithms for NP-complete problems
  - But maybe efficient algorithm still exists for testing the primality of a number – i.e., there are some problems that are NP, but not NP-complete

The Main Question/3

- Probably no, since:
  - Thousands of researchers have spent four decades in search of polynomial algorithms for many fundamental NP-complete problems without success
  - Consensus opinion: $P \neq NP$
- But maybe yes, since:
  - No success in proving $P \neq NP$ either

Dealing with NP-Completeness

- Hope that a worst case doesn't occur
  - Complexity theory deals with worst case behavior. The instance(s) you want to solve may be "easy"
    - TSP where all points are on a line or circle
    - 13,509 US city TSP problem solved (Cook et. al., 1998)
- Change the problem
  - Develop a heuristic, and hope it produces a good solution.
  - Design an approximation algorithm: algorithm that is guaranteed to find a high-quality solution in polynomial time
    - active area of research, but not always possible
- Keep trying to prove P = NP.

The Big Picture

- Summarizing: it is not known whether NP problems are tractable or intractable
- But, there exist provably intractable problems
  - Even worse – there exist problems with running times unimaginably worse than exponential!
- More bad news: there are **provably non-computable (undecidable)** problems
  - There are no (and there will not ever be) algorithms to solve these problems

Syllabus

1) Correctness and complexity of algorithms
2) Divide and conquer, recurrences
3) Sorting
4) Pointers, lists, sets, abstract data types
5) Trees, red-black trees
6) Hash tables
7) Dynamic programming
8) Graph algorithms

The Course – Syllabus/1

- **Data structures**
  - Simple data structures
    - *array, linked lists, stacks, queues, trees, heaps*
  - Dictionaries
    - *hash tables*
    - *binary search trees* (unbalanced)
    - *red-black trees*
  - *Graphs*
    - memory representation
    - DAG

The Course – Syllabus/2

- **Algorithmic techniques**
  - Divide and Conquer
    - *Merge sort, quicksort, binary search, ...*
  - Dynamic programming
    - *Matrix chain multiplication, longest common subsequence*
  - Search (graph) algorithms
    - *DFS, BFS*
    - *Prim, Kruskal, Dijkstra, Bellman*
    - *Toplogical sorting, SCC, Erdös*
  - Greedy algorithms
    - *Prim, Kruskal, Dijkstra*

The Course – Syllabus/3

- **Analyzing algorithms**
  - Correctness of algorithms
    - *pre/post-conditions*
    - *input/output*
    - *Invariants*
    - *case analysis*
  - Exact runtime
  - Asymptotic complexity
  - Recurrences

The Course – Syllabus/4

- **Constructing algorithms**
  - Break up into small pieces
  - ADT concept
  - Precise definitions of data structures and operations (signatures of procedures)
  - specification of input and output
  - Representative example of input and output

The Exam/1

- The exam will be written, you will have **two hours**. Roughly half an hour per exercise.
- Auxiliary material: **1 A4 sheet** with notes
- In the exam you have to **apply** things you learned during the course.
- Try to solve all problems. Do not get lost/stuck in details.

The Exam/2

- Template for solution:
  1) understand and explain problem (example)
  2) write down plan for solution; reduce to known techniques
  3) come up with algorithms
  4) analyze algorithms if required

The Exam/3

- *Good solution = simple and short solution that is understandable*
- *Being precise is important*
- *Solve step by step*

Preparation for the Exam

- To prepare for the exam:
  - Solve all assignments (do not start by studying/memorizing/learning an example solution)
  - Practice the first step
  - Do not only study the textbook (although you have to know the concepts).
  - Go to TAs or come to my office if you have questions

Thanks!

All the Best!!